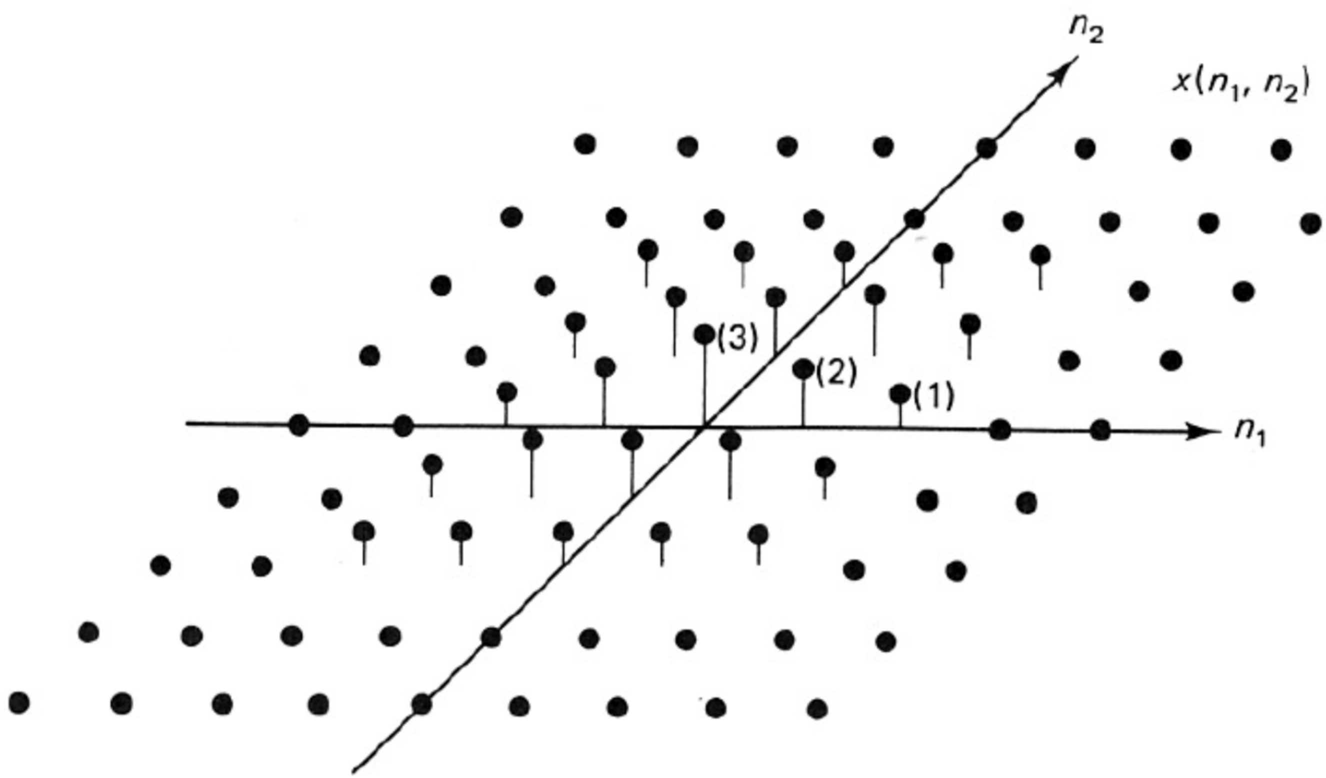
Video signals

IMAGE AS BIDIMENSIONAL SIGNALS

Image as a 2D sampling

A digital image can be considered as a 2D discrete signal

$$x(n_1, n_2)$$



Impulse definition

$$\delta(n_1,n_2) \triangleq \begin{cases} 1 & \text{for } (n_1,n_2) = (0,0), \\ 0 & \text{for } (n_1,n_2) \neq (0,0), \end{cases}$$

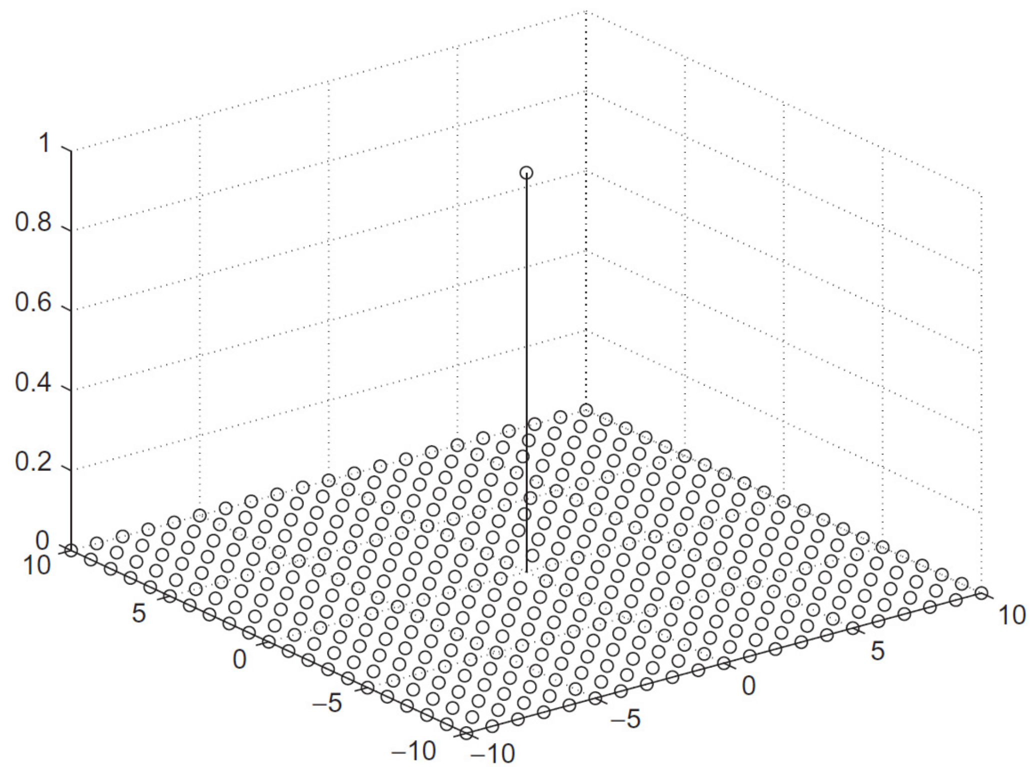
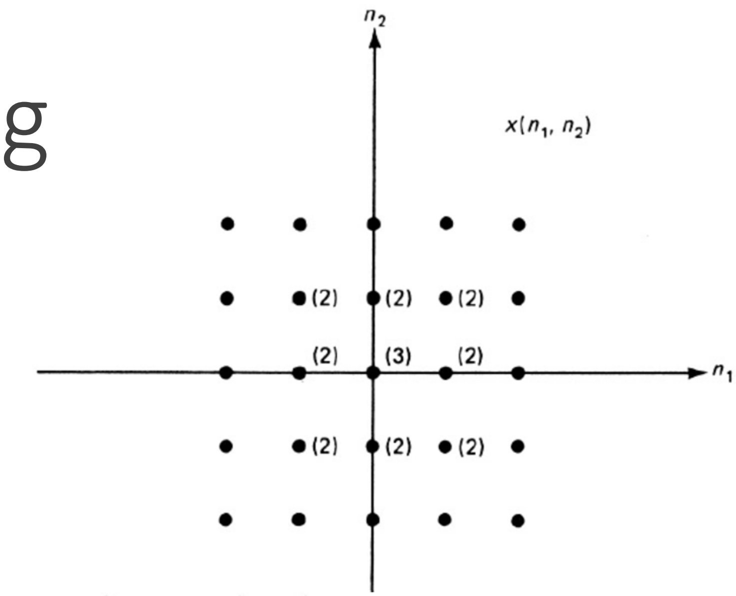


Image as a 2D sampling

Each sequence can be considered as a sequence of impulses

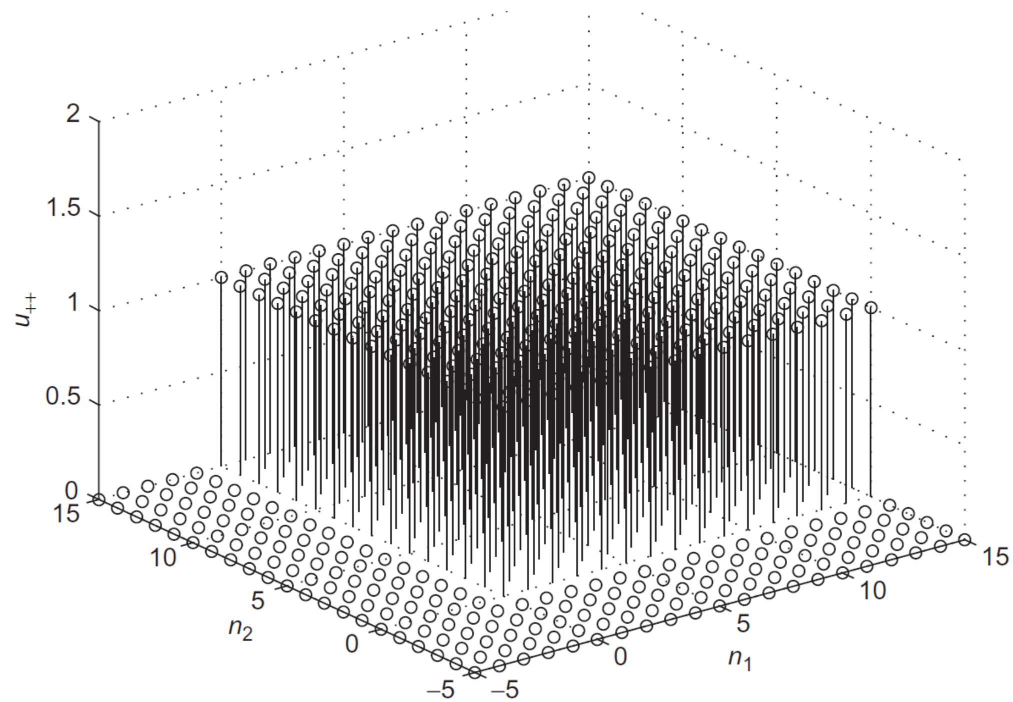


$$
\begin{aligned}
x(n_1,n_2) = \cdots &+ x(-1,-1)\,\delta(n_1+1,n_2+1) + x(0,-1)\,\delta(n_1,n_2+1) \\
&+ x(1,-1)\,\delta(n_1-1,n_2+1) + \cdots + x(-1,0)\,\delta(n_1+1,n_2) \\
&+ x(0,0)\,\delta(n_1,n_2) + x(1,0)\,\delta(n_1-1,n_2) \\
&+ \cdots + x(-1,1)\,\delta(n_1+1,n_2-1) \\
&+ x(0,1)\,\delta(n_1,n_2-1) + x(1,1)\,\delta(n_1-1,n_2-1) + \cdots \\
= \sum_{k_1=-\infty}^{\infty} &\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\,\delta(n_1-k_1,n_2-k_2).
\end{aligned}
$$

The unit step function

The step function as a combination of impulses.

$$u(n_1, n_2) \triangleq \begin{cases} 1, & n_1 \geq 0, n_2 \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$



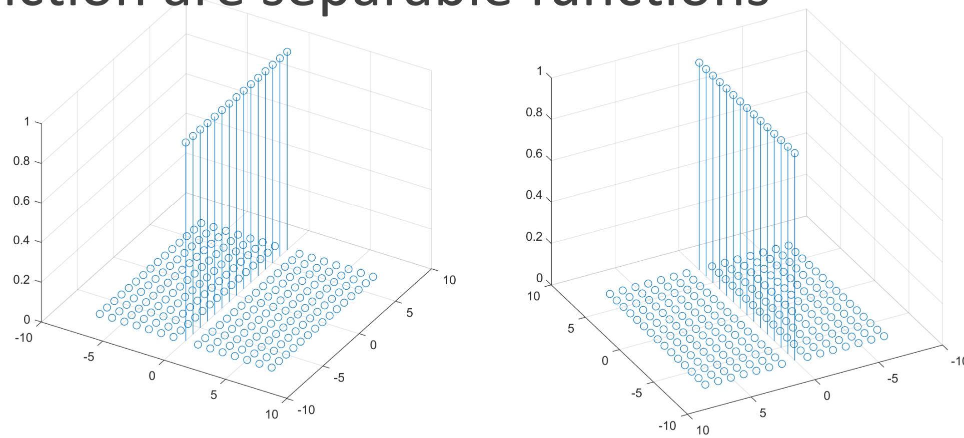
Separable sequences

A separable 2D sequence can be written as:

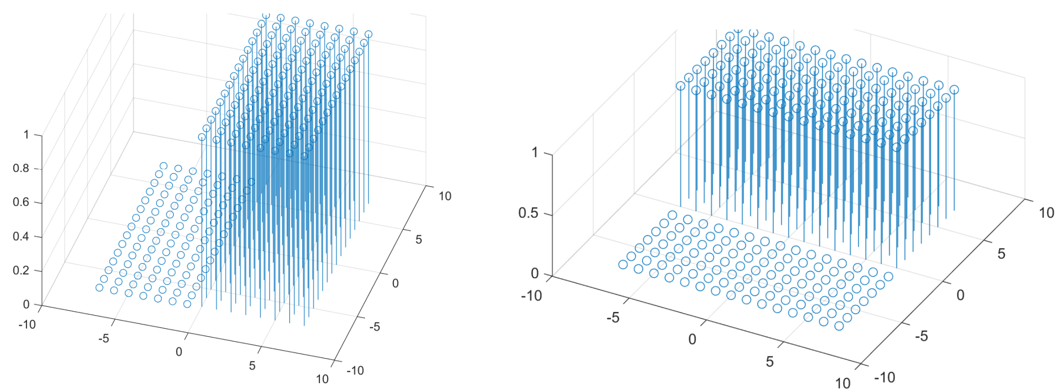
$$x(n_1, n_2) = x_1(n_1)x_2(n_2) \quad \text{for all } n_1 \text{ and } n_2,$$

The impulse and step function are separable functions

$$\delta(n_1, n_2) = \delta(n_1)\,\delta(n_2)$$



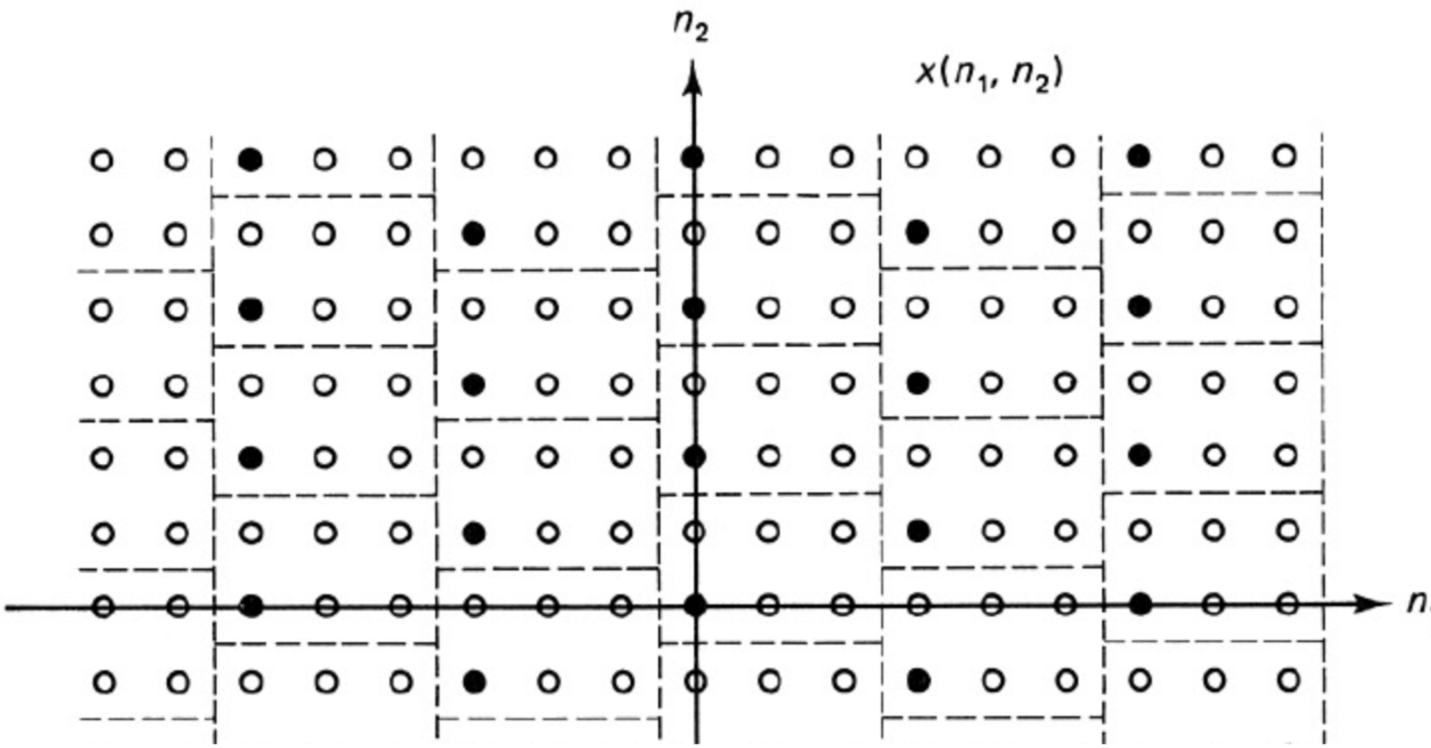
$$u(n_1, n_2) = u(n_1)u(n_2)$$



Periodic sequences

A sequence $x(n_1, n_2)$ is periodic of period N_1 x N_2 if:

$$x(n_1, n_2) = x(n_1 + N_1, n_2) = x(n_1, n_2 + N_2) \quad \text{for all } (n_1, n_2)$$



Linear Shift Invariant Systems

Linearity

$$T[ax_1(n_1, n_2) + bx_2(n_1, n_2)] = ay_1(n_1, n_2) + by_2(n_1, n_2)$$

Spatial invariance

$$T[x(n_1 - m_1, n_2 - m_2)] = y(n_1 - m_1, n_2 - m_2)$$

The impulse response

$$y(n_1, n_2) = T[x(n_1, n_2)] = T\left[\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\,\delta(n_1 - k_1, n_2 - k_2)\right]$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) T[\delta(n_1 - k_1, n_2 - k_2)].$$

Convolution

Defined the impulse response

$$h(n_1, n_2) = T[\delta(n_1, n_2)].$$

The Input/Output relation is given by:

$$y(n_1, n_2) = T[x(n_1, n_2)] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)h(n_1 - k_1, n_2 - k_2).$$

$$y(n_1, n_2) = x(n_1, n_2) * h(n_1, n_2)$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)h(n_1 - k_1, n_2 - k_2).$$

Convolution properties

Commutativity

$$x(n_1, n_2) * y(n_1, n_2) = y(n_1, n_2) * x(n_1, n_2)$$

Associativity

$$(x(n_1, n_2) * y(n_1, n_2)) * z(n_1, n_2) = x(n_1, n_2) * (y(n_1, n_2) * z(n_1, n_2))$$

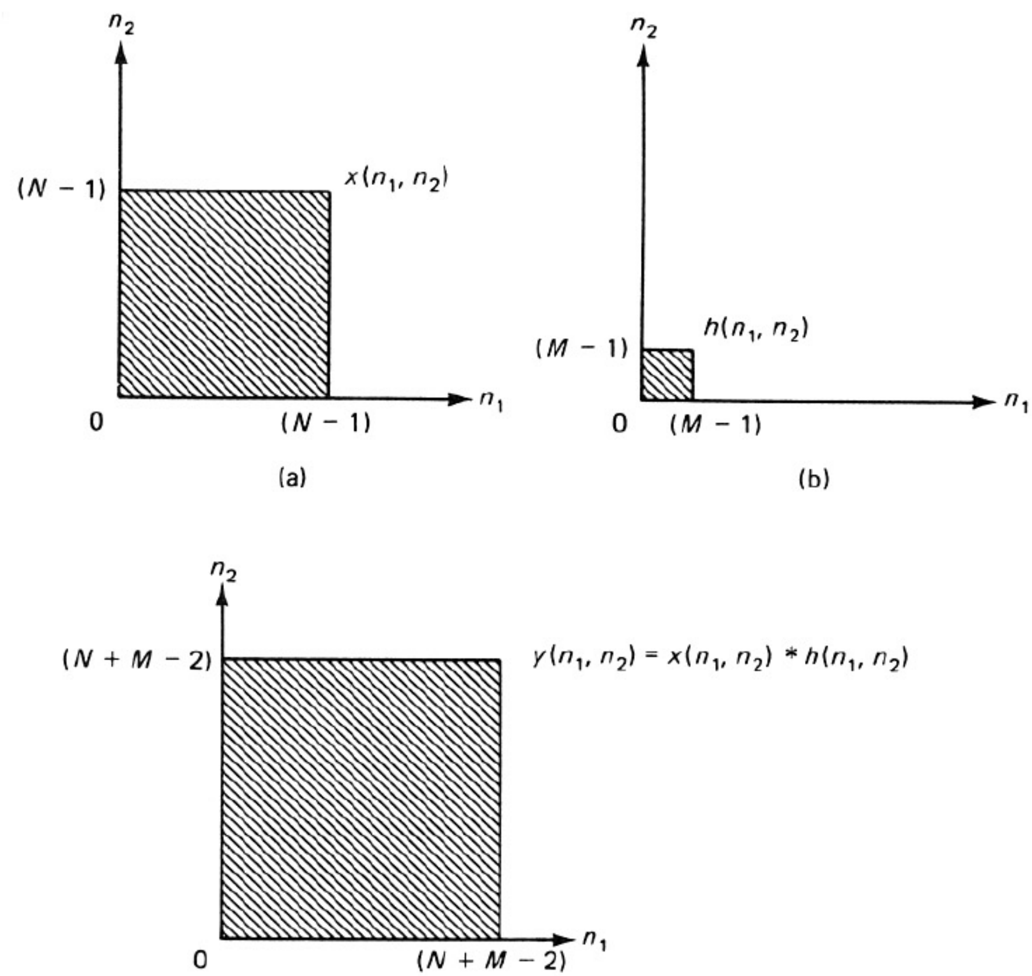
Distributivity

$$x(n_1, n_2) * (y(n_1, n_2) + z(n_1, n_2)) = (x(n_1, n_2) * y(n_1, n_2)) + (x(n_1, n_2) * z(n_1, n_2))$$

Convolution with Shifted Impulse

$$x(n_1, n_2) * \delta(n_1 - m_1, n_2 - m_2) = x(n_1 - m_1, n_2 - m_2)$$

Convolution examples



The 2D Fourier Transform

The analysis and synthesis formulas for the 2D continuous Fourier transform are as follows:

Analysis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy$$

Synthesis

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}du\,dv$$

Separability of 2D Fourier Transform

The 2D analysis formula can be written as a 1D analysis in the *x* direction followed by a *1D* analysis in the *y* direction*:*

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi ux} dx \right] e^{-j2\pi vy} dy$$

The 2D synthesis formula can be written as a 1D synthesis in the *x* direction followed by a 1D synthesis in *y* direction:

$$f(x,y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(u,v) e^{j2\pi ux} du \right] e^{j2\pi vy} dv$$

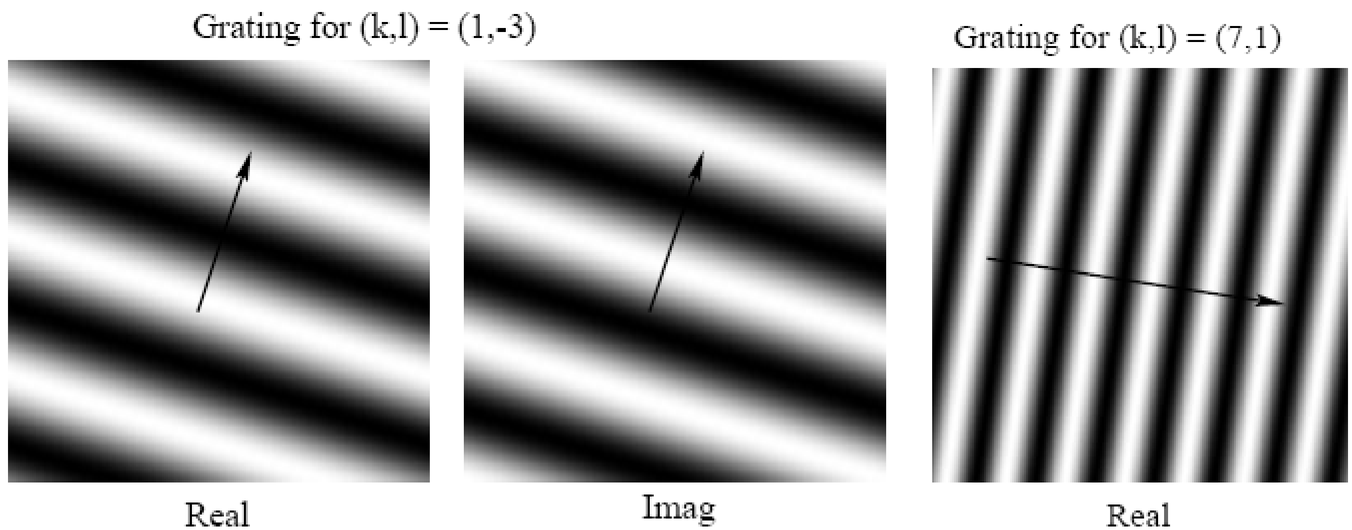
Separability Theorem

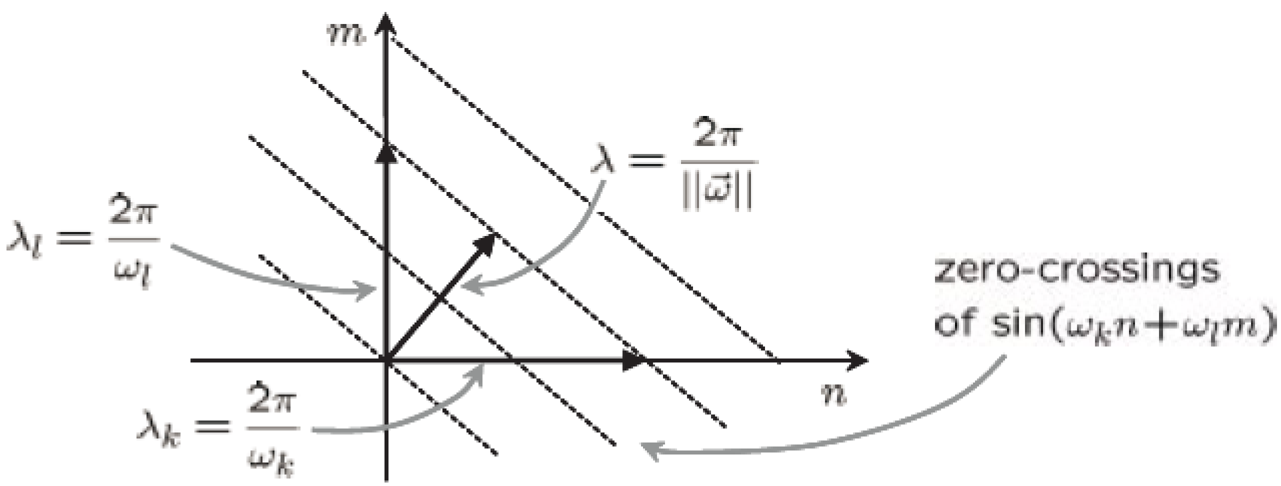
$$f(x,y) = f(x)g(y) \xrightarrow{\mathcal{F}} F(u,v) = F(u)G(v)$$

Proof:

$$\begin{aligned} F(u,v) &\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)e^{-j2\pi ux}e^{-j2\pi vy}\,dx\,dy \\ &= \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}\,dx \int_{-\infty}^{\infty} g(y)e^{-j2\pi vy}dy \\ &= F(u)\,G(v) \end{aligned}$$

2D Fourier Basis Functions





The 2D Discrete Fourier Transform for periodic signals

The analysis and synthesis formulas for the 2D discrete Fourier transform are as follows:

Analysis

$$\hat{F}(k,\ell) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} F(m,n) e^{-j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

$$F(m,n) = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{\ell=0}^{N-1} \hat{F}(k,\ell) e^{j2\pi\left(k\frac{m}{M}+\ell\frac{n}{N}\right)}$$

Separability of 2D DFT

$$\hat{F}(k,\ell) =$$

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\left[\frac{1}{\sqrt{M}}\sum_{m=0}^{M-1}F(m,n)e^{-j2\pi\left(k\frac{m}{M}\right)}\right]e^{-j2\pi\left(\ell\frac{n}{N}\right)}$$

The 2D forward DFT (for a squared input matrix of size SxS)can be written in matrix notation:

$$\hat{\mathbf{F}} = \left(\mathbf{W}^{*}\mathbf{F}\right)\mathbf{W}^{*}$$

Where *r* and *c* are row and column indexes starting from zero.

$$W_{rc}^{*} = \frac{1}{\sqrt{S}}e^{-j2\pi\frac{rc}{S}}$$

Separability of 2D DFT

And

$$F(m,n) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \left[\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \hat{F}(k,\ell) e^{j2\pi\left(k\frac{m}{M}\right)} \right] e^{j2\pi\left(\ell\frac{n}{N}\right)}.$$

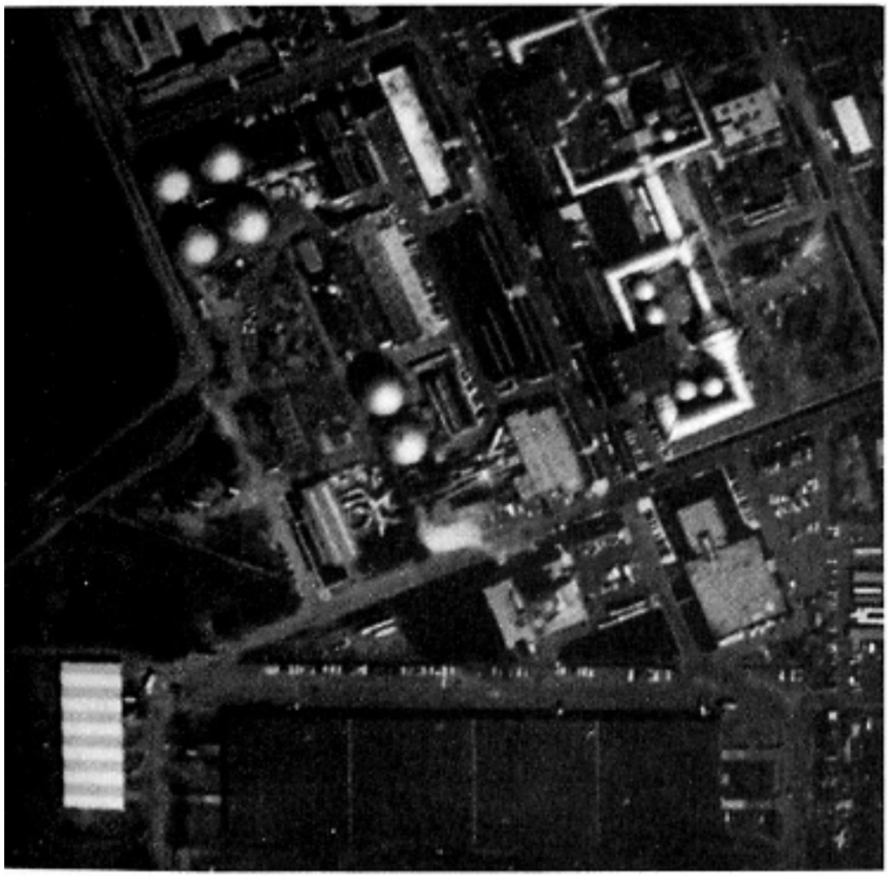
The 2D inverse DFT can be written in matrix notation:

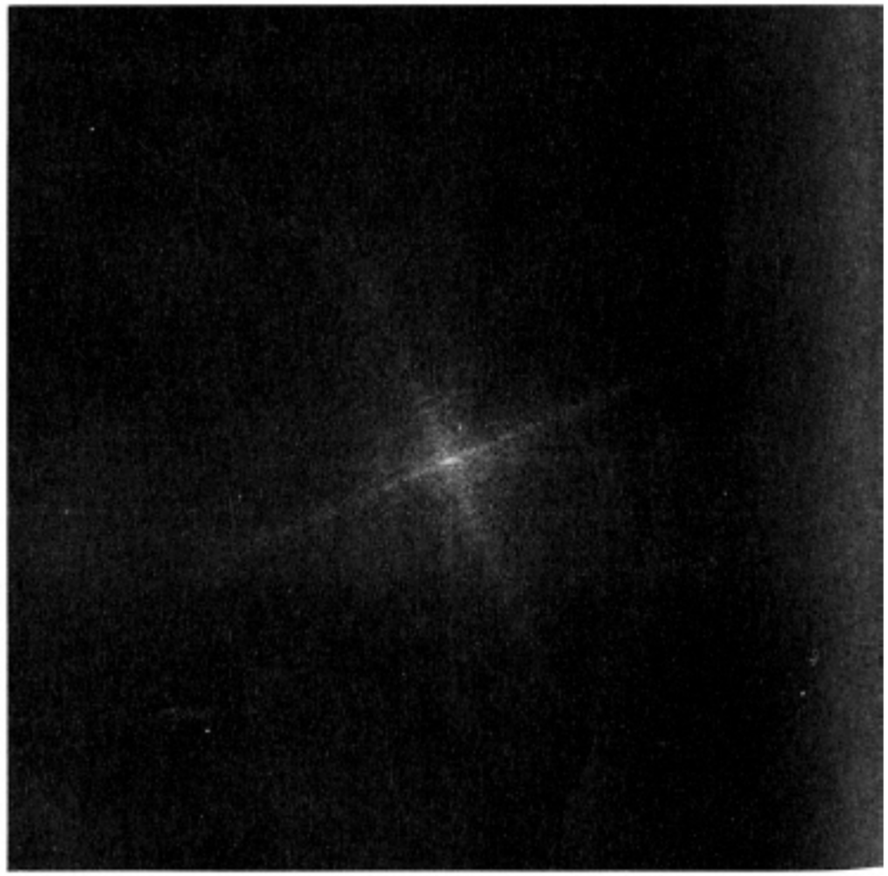
$$\mathbf{F} = \left(\mathbf{W}\hat{\mathbf{F}}\right)\mathbf{W}$$

Where the matrix elements are

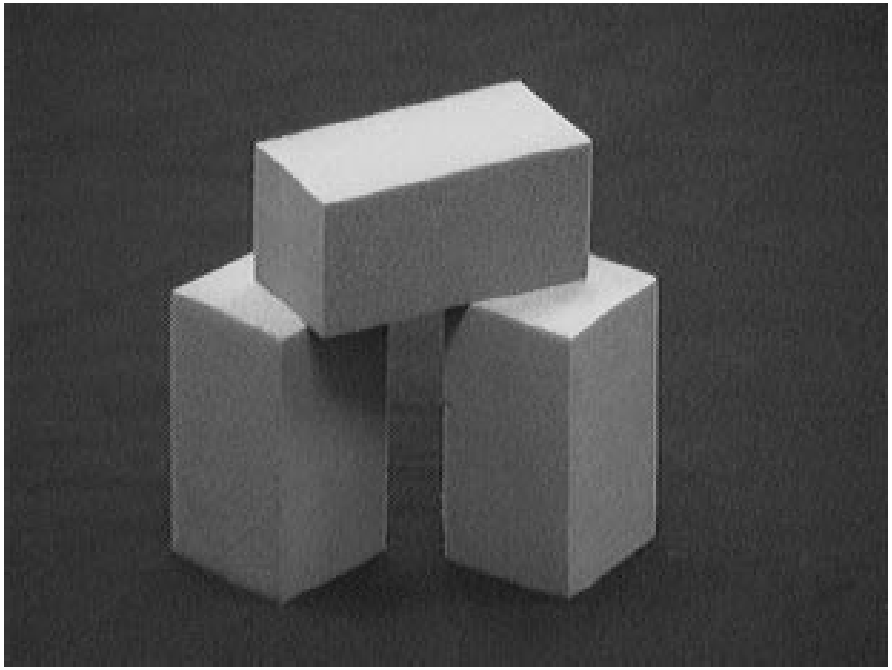
$$W_{rc} = \frac{1}{\sqrt{A}} e^{j2\pi\frac{rc}{A}}$$

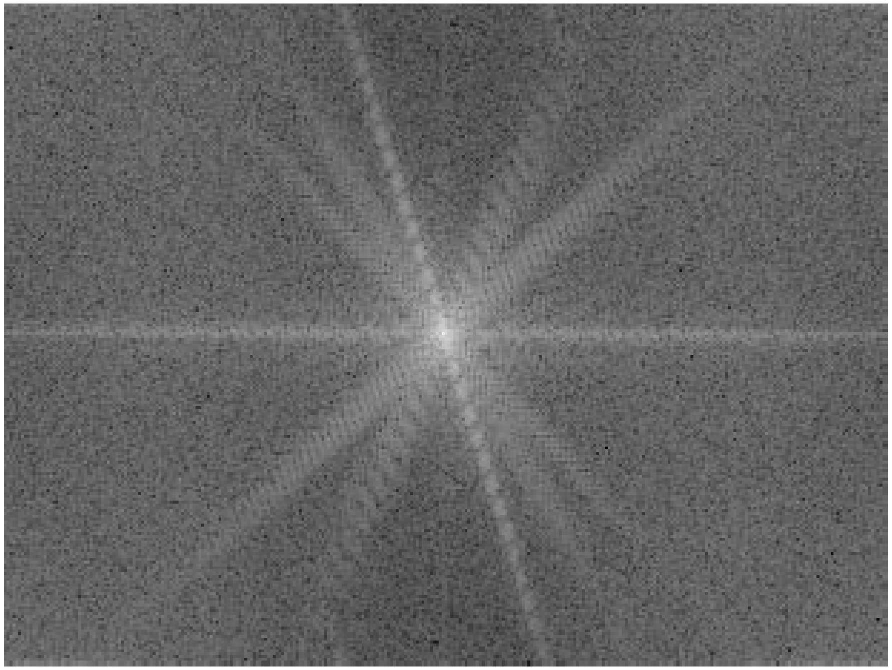
Transform examples



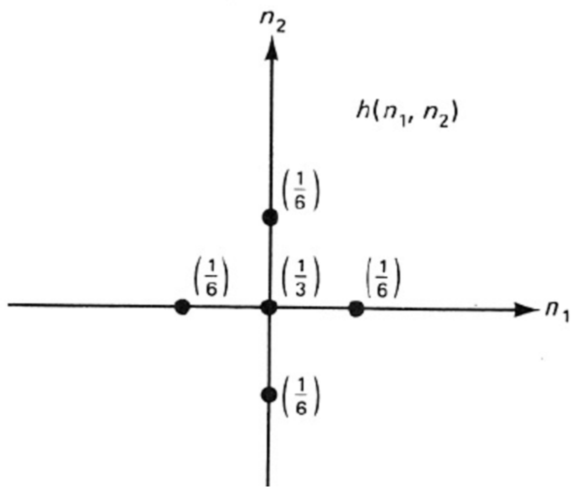


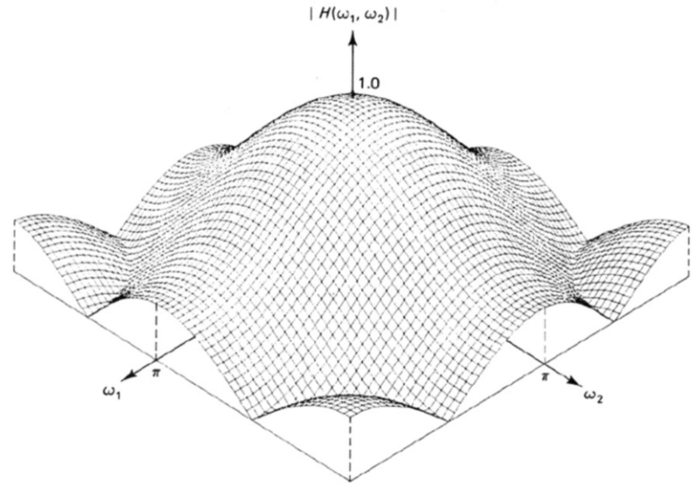
2D DFT Example

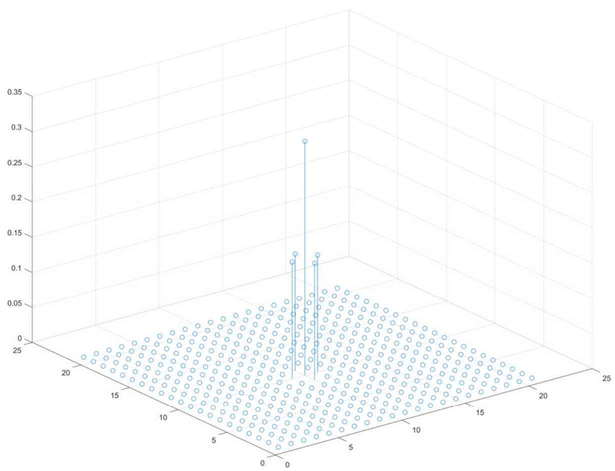




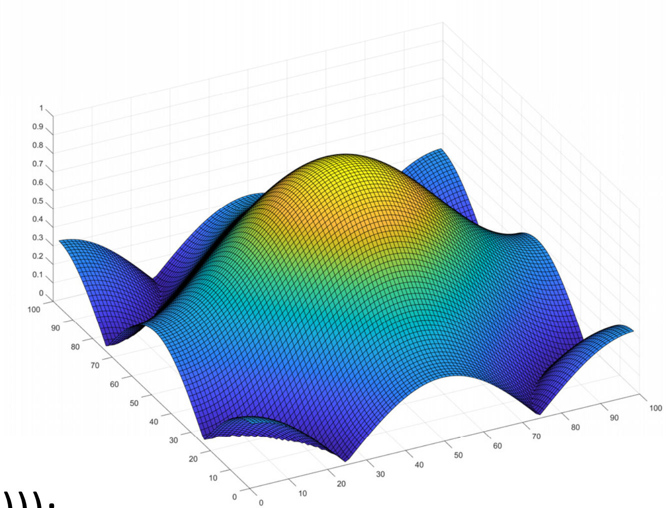
DFT example



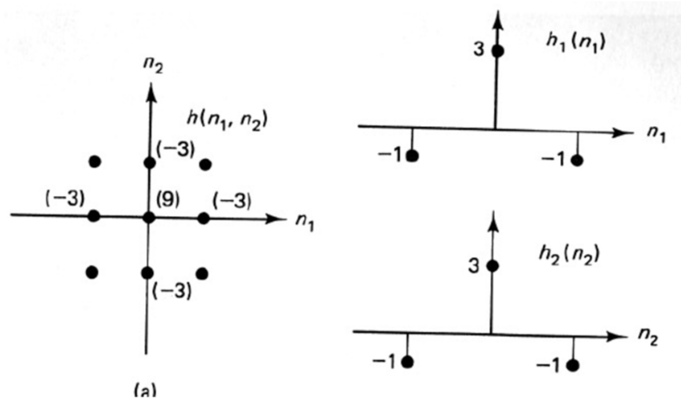


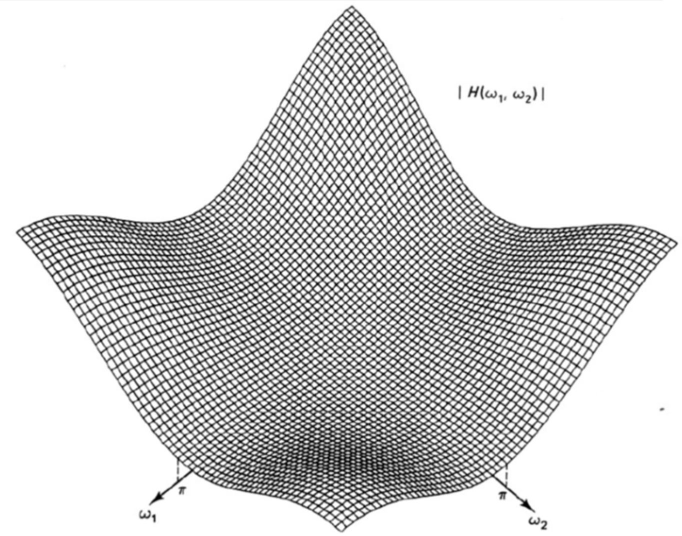


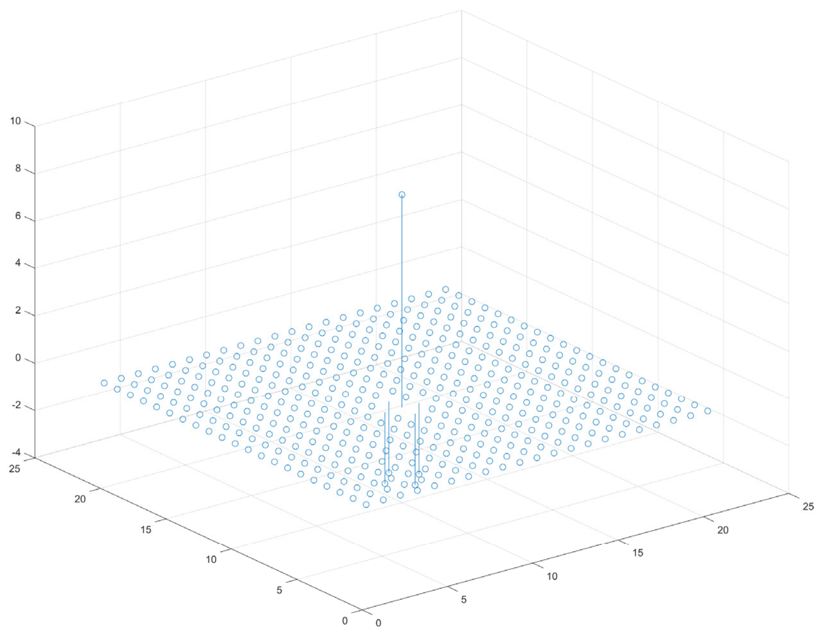
```
h=zeros(99);
h(51,51)=1/3;
h(50,51)=1/6;
h(52,51)=1/6;
h(51,52)=1/6;
h(51,50)=1/6;
H=fft2(h);
surf(abs(fftshift(H)));
```



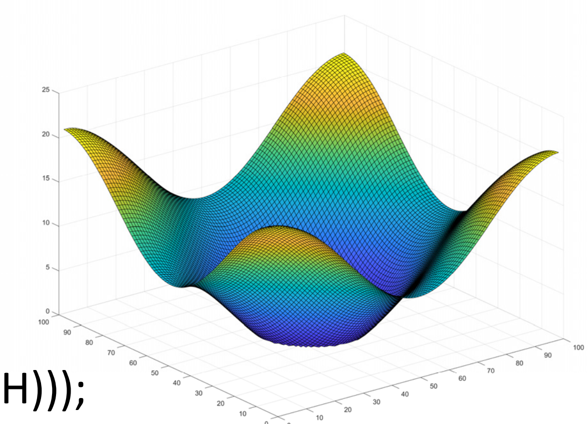
Hi-Pass Filter example



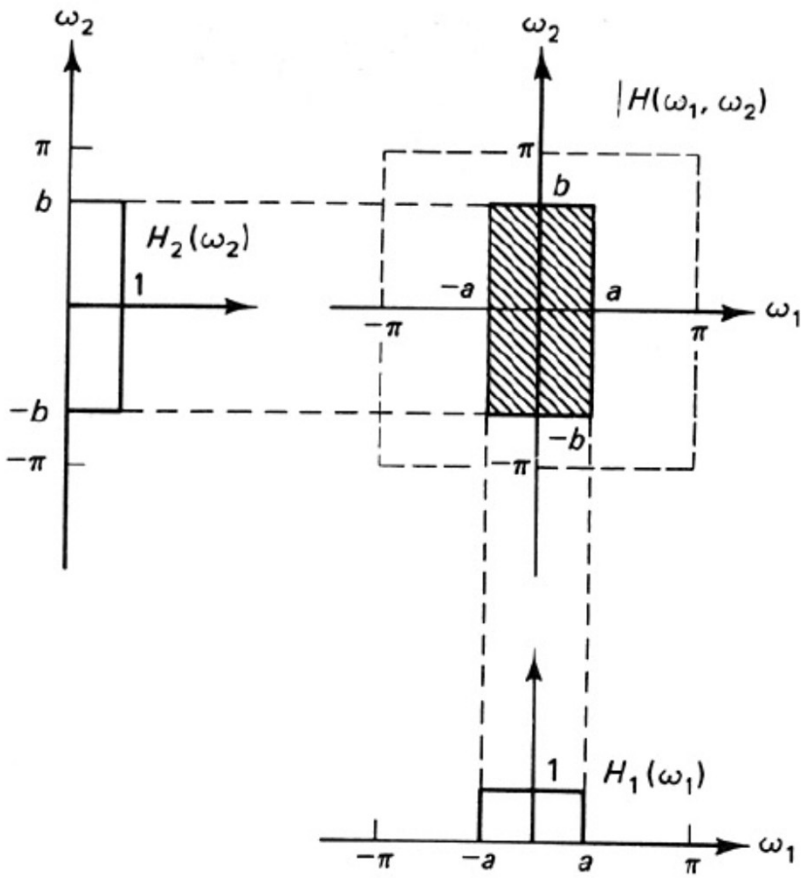




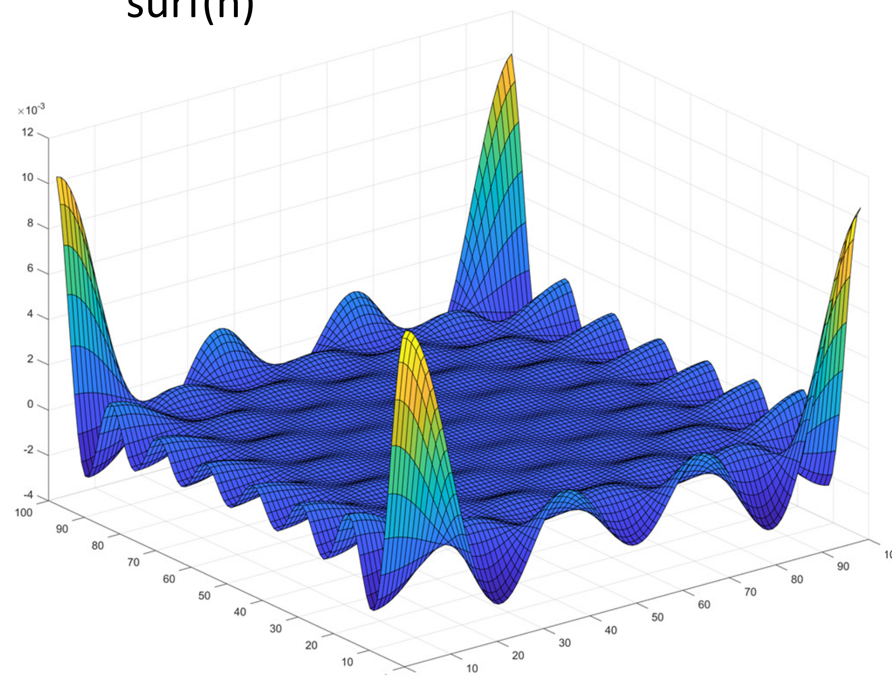

```
h=zeros(99);
h(51,51)=9;
h(50,51)=-3;
h(52,51)=-3;
h(51,50)=-3;
h(51,52)=-3;
H=fft2(h);
surf(abs(fftshift(H)));
```



Separable Low-pass Filter example



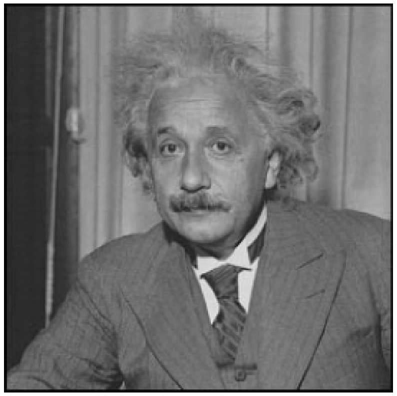
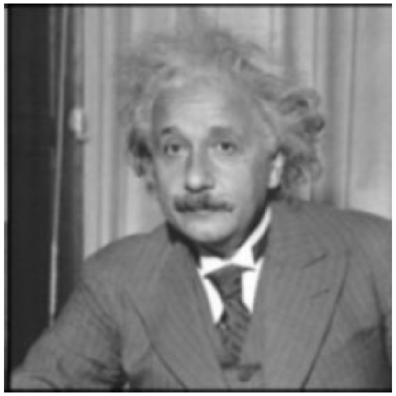

```
H=zeros(99);a=3;b=7;
H(51-b:51+b,51-a:51+a)=1;
h=ifft2(H);
surf(h)
```

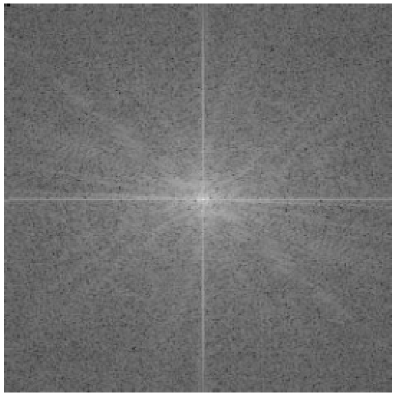


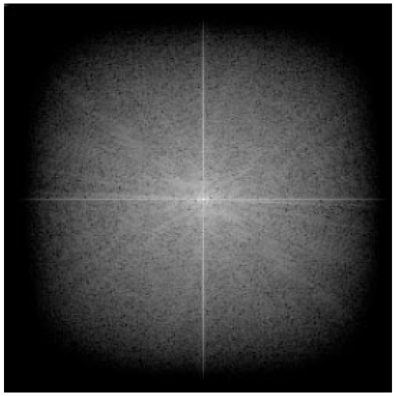
$$h(n_1, n_2) = h_1(n_1)h_2(n_2) = \frac{\sin an_1}{\pi n_1}\frac{\sin bn_2}{\pi n_2}$$

low-pass (blurred) version of an image

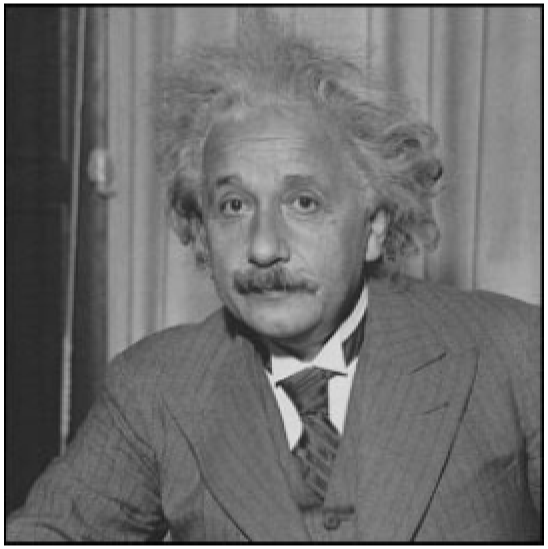
h(n)= $\frac{1}{16}(1,4,6,4,1)$ (1D impulse response in both dimensions)

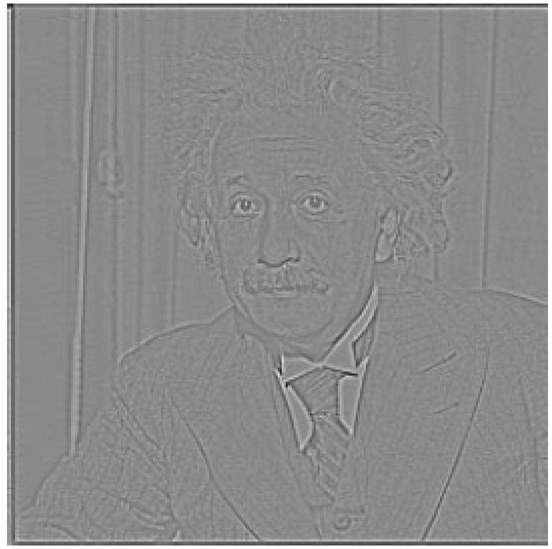



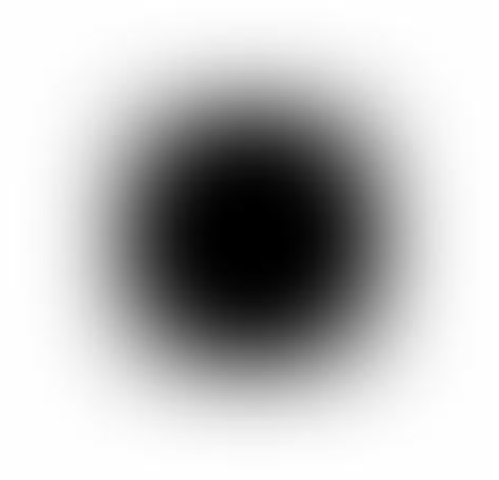




Hi-Pass Filter







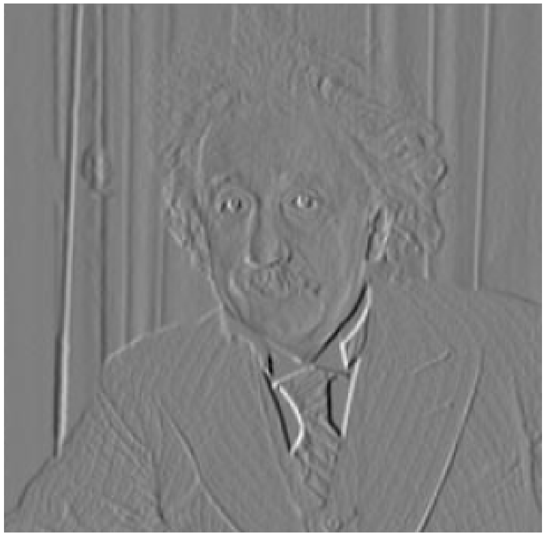
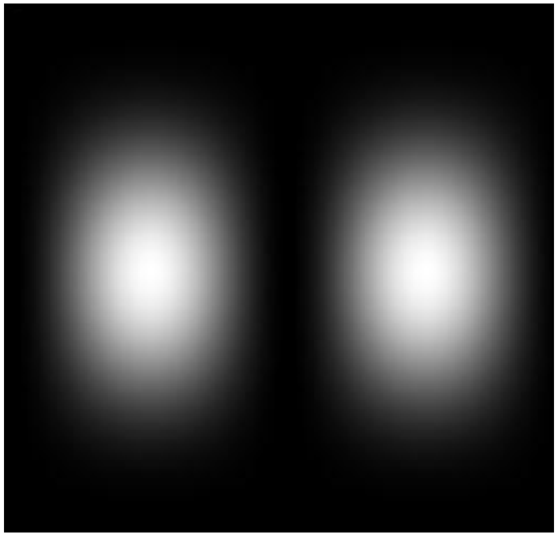
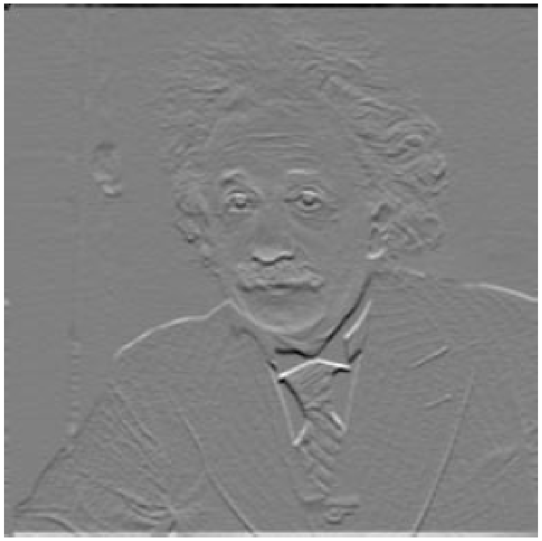
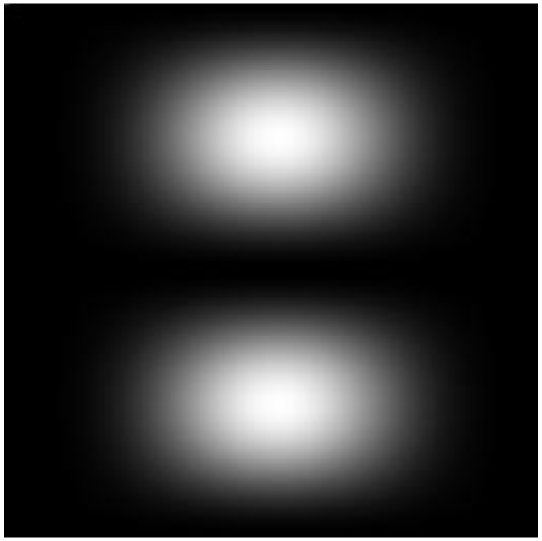
impulse response is defined by $\delta(n)$-$h(n,m)$ where $h(n,m)$ is the separable blurring kernel used in the previous figure

Band Pass Filter

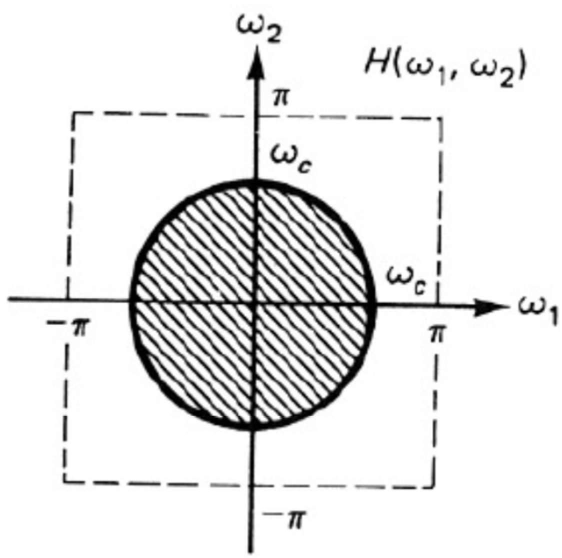
a band-pass filtered version of Albert, and the amplitude spectrum of the filter.

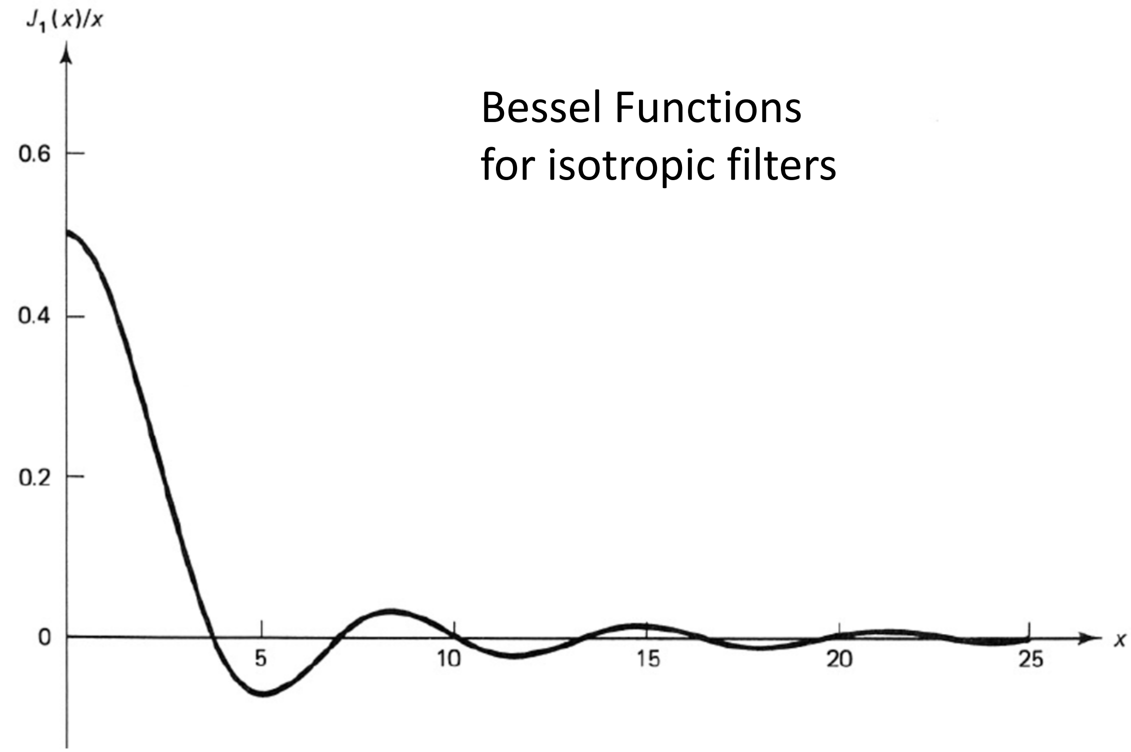
This impulse response is defined by the difference of two low-pass filters.

Directional filters

Circular filter example





Bessel Functions
for isotropic filters

$$h(n_1, n_2) = \frac{\omega_C}{2\pi\sqrt{n_1^2 + n_2^2}} J_1(\omega_C \sqrt{n_1^2 + n_2^2})$$

Nyquist Sampling Theorem and Aliasing

Consider a perspective image of an infinite checkerboard.

The signal is dominated by high frequencies in the image near the horizon.

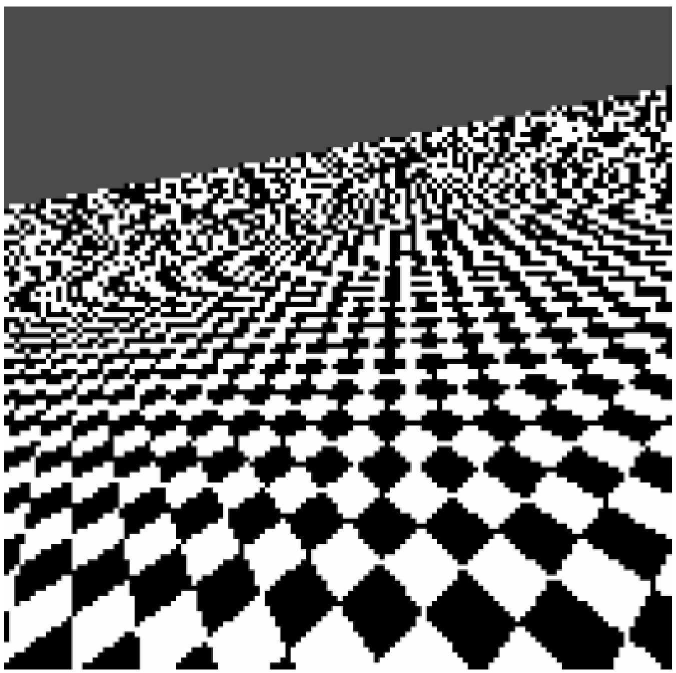
Properly designed cameras blur the signal before sampling using:

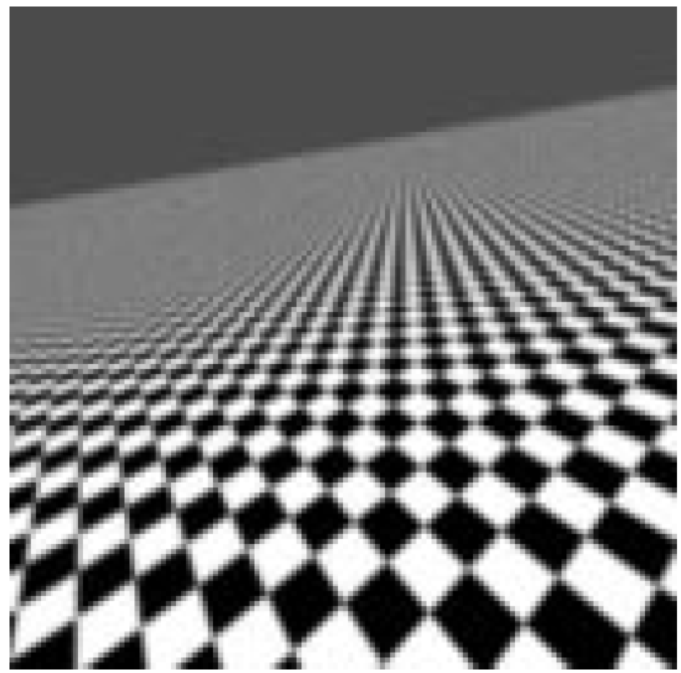
- The point spread function due to diffraction
- Imperfect focus
- Averaging the signal over each CCD element.

Nyquist Sampling Theorem and Aliasing

These operations attenuate high frequency components in the signal.

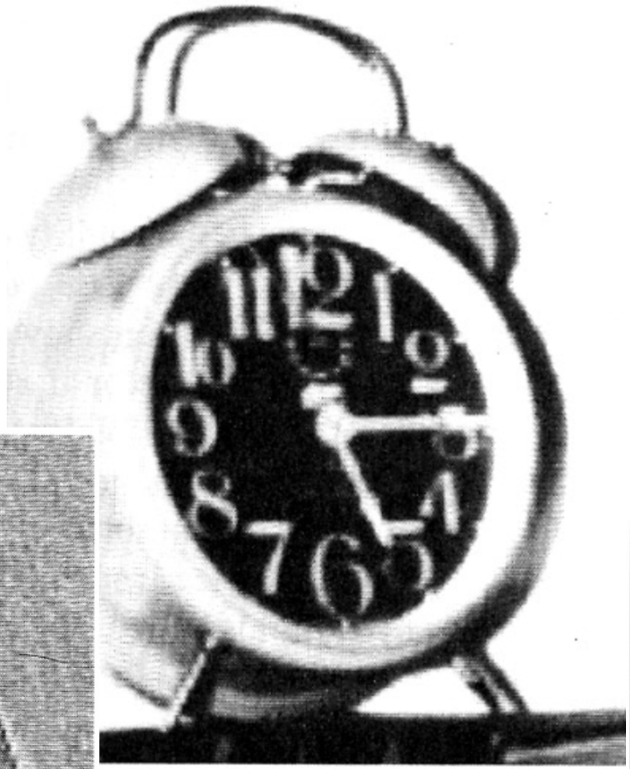
Without this (physical) preprocessing, the sampled image can be severely aliased (corrupted):





Reconstruction using just phase or intensity

Only phase



Only amplitude

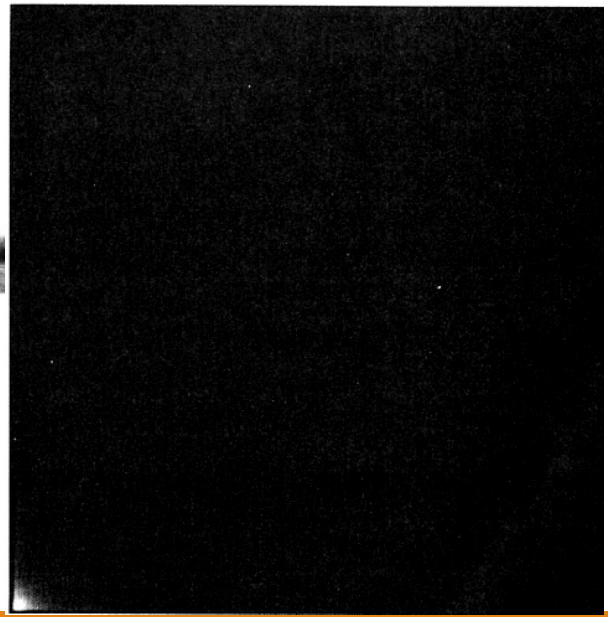


Image superposition combining phases and intesities