Fourier Transform Properties (Continuous-Time Signals)

Main properties and proofs

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Let

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$$

denote the Fourier Transform pair, with definition

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt.$$

1 Linearity

Property:

$$ax_1(t) + bx_2(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad aX_1(f) + bX_2(f)$$

Proof: Directly from the linearity of the integral.

2 Time Shifting

Property:

$$x(t - t_0) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad e^{-j2\pi f t_0} X(f)$$

Proof:

$$\begin{aligned}
\mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0) e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f(\tau+t_0)} d\tau \\
&= e^{-j2\pi f t_0} X(f).
\end{aligned}$$

3 Frequency Shifting (Modulation)

Property:

$$x(t)e^{j2\pi f_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(f - f_0)$$

Proof:

$$\begin{aligned}
\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} &= \int x(t) e^{j2\pi f_0 t} e^{-j2\pi ft} dt \\
&= \int x(t) e^{-j2\pi (f-f_0)t} dt = X(f - f_0).
\end{aligned}$$

4 Scaling

Property:

$$x(at) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tfrac{1}{|a|} X\left(\tfrac{f}{a}\right), \quad a \neq 0$$

Proof:

$$\begin{aligned}
\mathcal{F}\{x(at)\} &= \int x(at) e^{-j2\pi ft} dt \\
&= \tfrac{1}{|a|} \int x(\tau) e^{-j2\pi f\tau/a} d\tau \\
&= \tfrac{1}{|a|} X\left(\tfrac{f}{a}\right).
\end{aligned}$$

5 Time Differentiation

Property:

$$\frac{d}{dt}x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad j2\pi f X(f)$$

Proof: By integration by parts (assuming vanishing boundary terms).

6 Frequency Differentiation

Property:

$$(-j2\pi t)x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{d}{df}X(f)$$

Proof:

$$\begin{aligned}\frac{d}{df}X(f) &= \frac{d}{df}\int x(t)e^{-j2\pi ft}dt \\ &= \int x(t)(-j2\pi t)e^{-j2\pi ft}dt.\end{aligned}$$

7 Convolution in Time

Property:

$$(x_1 * x_2)(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_1(f)X_2(f)$$

Proof: Let

$$y(t) = (x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau.$$

Compute the Fourier transform $Y(f) = \mathcal{F}\{y(t)\}$:

$$\begin{aligned}Y(f) &= \int_{-\infty}^{\infty} y(t)\, e^{-j2\pi ft}\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-j2\pi ft}\, dt.\end{aligned}$$

Under the usual hypotheses that permit changing the order of integration (e.g. $x_1, x_2 \in L^1$ or by Tonelli/Fubini), we swap the integrals:

$$Y(f) = \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi ft}\, dt \right) d\tau.$$

In the inner integral perform the change of variable $u = t - \tau$ (so $t = u + \tau$, $dt = du$):

$$\begin{aligned}\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j2\pi ft}\, dt &= \int_{-\infty}^{\infty} x_2(u)\, e^{-j2\pi f(u+\tau)}\, du \\ &= e^{-j2\pi f\tau} \int_{-\infty}^{\infty} x_2(u)\, e^{-j2\pi fu}\, du \\ &= e^{-j2\pi f\tau}\, X_2(f).\end{aligned}$$

Insert this back into $Y(f)$:

$$Y(f) = X_2(f) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j2\pi f\tau}\, d\tau = X_2(f)\, X_1(f).$$

Therefore

$$\boxed{\mathcal{F}\{x_1 * x_2\}(f) = X_1(f)\, X_2(f)},$$

with the usual proviso that the convolutions/integrals are well-defined (or in the sense of tempered distributions).

8 Multiplication in Time

Property:

$$x_1(t)x_2(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X_1(f) * X_2(f)$$

Proof:

$$\mathcal{F}\{x_1(t)\, x_2(t)\}(f) = (X_1 * X_2)(f) = \int_{-\infty}^{\infty} X_1(f-\nu)\, X_2(\nu)\, d\nu.$$

Start from the Fourier transform of the product:

$$Y(f) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j2\pi f t}\, dt.$$

Assume x_2 admits the inverse Fourier representation $x_2(t) = \int_{-\infty}^{\infty} X_2(\nu)\, e^{j2\pi\nu t}\, d\nu$ (which holds for sufficiently nice signals or in the distributional sense). Substitute into $Y(f)$:

$$Y(f) = \int_{-\infty}^{\infty} x_1(t) \left(\int_{-\infty}^{\infty} X_2(\nu)\, e^{j2\pi\nu t}\, d\nu \right) e^{-j2\pi f t}\, dt.$$

Under hypotheses allowing interchange of integrals (Fubini/Tonelli), swap the order:

$$Y(f) = \int_{-\infty}^{\infty} X_2(\nu) \left(\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi(f-\nu)t}\, dt \right) d\nu.$$

Recognize the inner integral as $X_1(f-\nu)$:

$$\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi(f-\nu)t}\, dt = X_1(f-\nu).$$

Hence

$$Y(f) = \int_{-\infty}^{\infty} X_2(\nu)\, X_1(f-\nu)\, d\nu = \int_{-\infty}^{\infty} X_1(f-\nu)\, X_2(\nu)\, d\nu,$$

which is precisely the convolution $X_1 * X_2$ evaluated at f. Therefore

$$\boxed{\mathcal{F}\{x_1(t)\, x_2(t)\}(f) = (X_1 * X_2)(f).}$$

Remark. The derivation is valid under the usual integrability or distributional assumptions that justify the inverse transform representation of x_2 and the interchange of integrals.

9 Parseval's Theorem

Property:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Proof: From bilinear identity

$$\int x(t)\overline{y(t)} dt = \int X(f)\overline{Y(f)} df.$$

Taking $y = x$ yields Parseval.

Proof of the Bilinear Identity: Start from the inverse Fourier representation of x:

$$x(t) = \int_{-\infty}^{\infty} X(\nu)\, e^{j2\pi\nu t}\, d\nu.$$

Compute the inner product:

$$\int_{-\infty}^{\infty} x(t)\, \overline{y(t)}\, dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X(\nu)\, e^{j2\pi\nu t}\, d\nu \right) \overline{y(t)}\, dt.$$

Assuming conditions that allow exchange of integrals (Fubini/Tonelli), we swap the order:

$$\int_{-\infty}^{\infty} x(t)\,\overline{y(t)}\,dt = \int_{-\infty}^{\infty} X(\nu)\left(\int_{-\infty}^{\infty} e^{j2\pi\nu t}\,\overline{y(t)}\,dt\right)d\nu.$$

The inner integral can be recognized as

$$\int_{-\infty}^{\infty} e^{j2\pi\nu t}\,\overline{y(t)}\,dt = \overline{\int_{-\infty}^{\infty} y(t)\,e^{-j2\pi\nu t}\,dt} = \overline{Y(\nu)}.$$

Thus,

$$\int_{-\infty}^{\infty} x(t)\,\overline{y(t)}\,dt = \int_{-\infty}^{\infty} X(\nu)\,\overline{Y(\nu)}\,d\nu,$$

which is the desired bilinear identity.