The Z-Transform

Lesson 3

The z-transform

The z-transform of a sequence $x[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The z-transform can also be thought of as an operator $\mathcal{Z}\{\cdot\}$ that transforms a sequence to a function:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z).$$

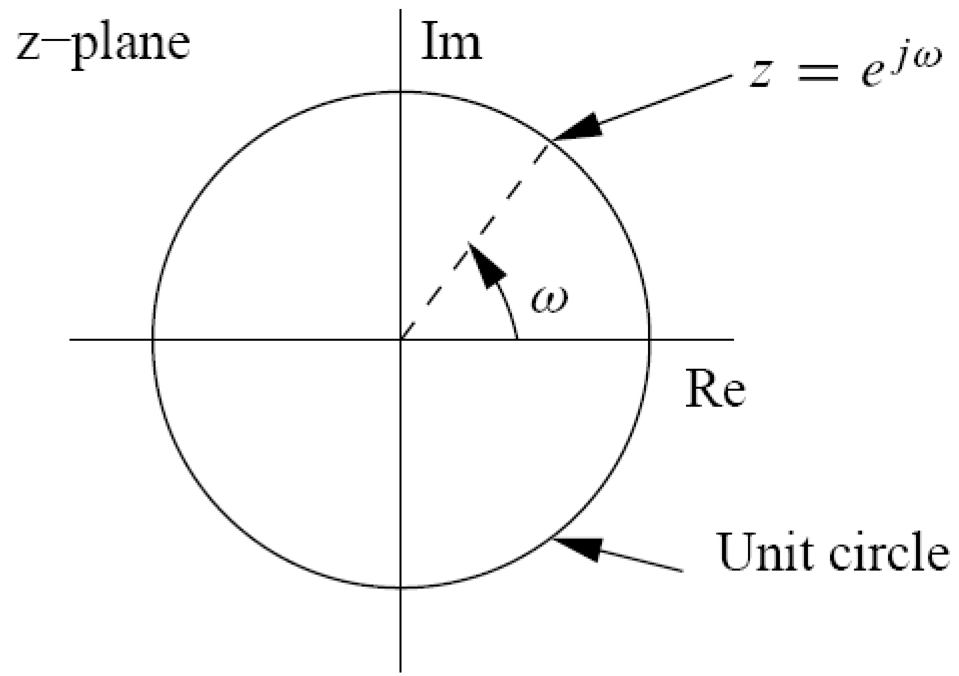
In both cases z is a continuous complex variable.

We may obtain the DTFT from the z-transform by making the substitution $z = e^{j\omega}$. This corresponds to restricting $|z| = 1$. Also, with $z = re^{j\omega}$,

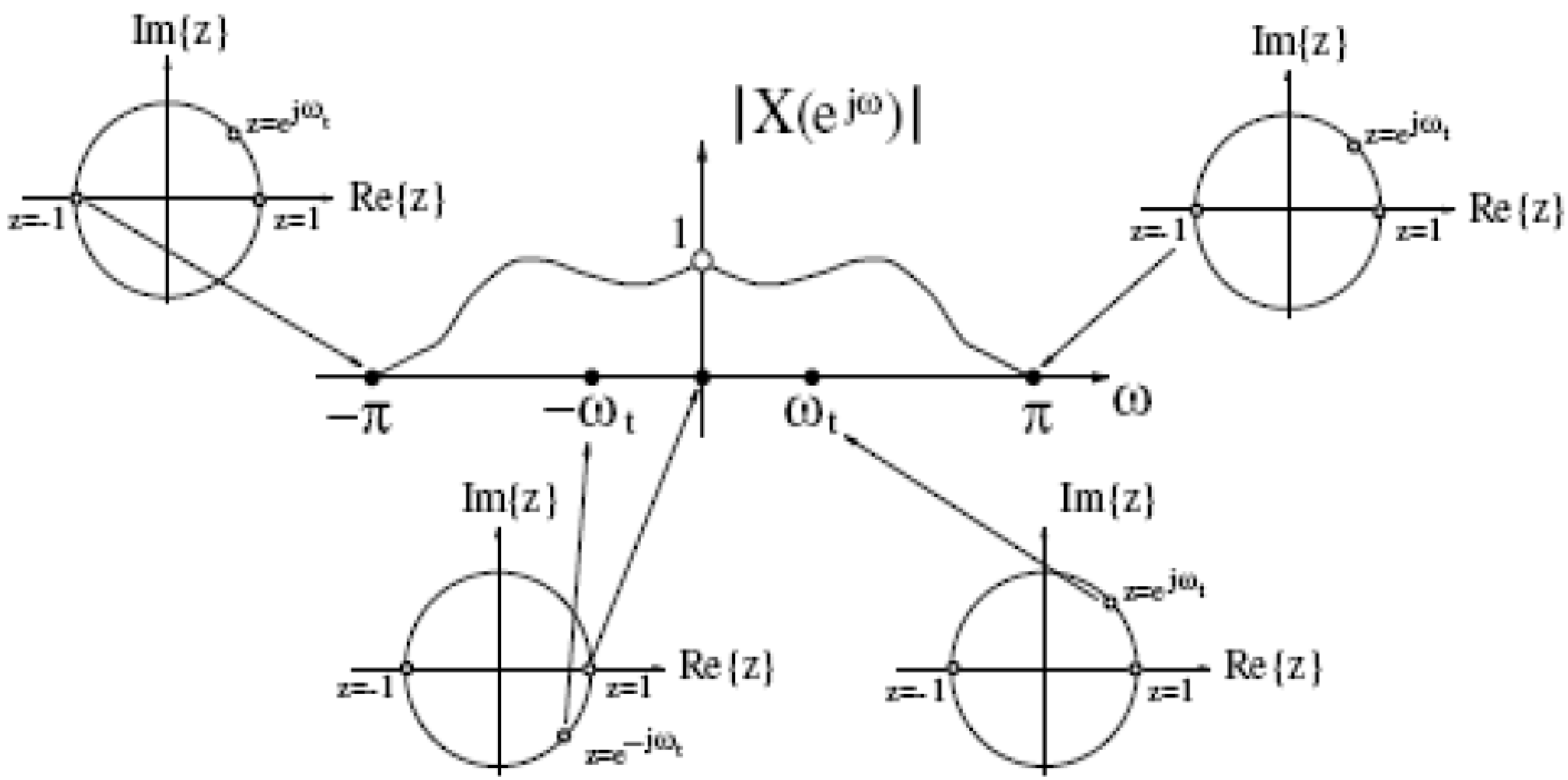
Fourier and z transform

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}.$$

That is, the z-transform is the DTFT of the sequence $x[n]r^{-n}$. For $r = 1$ this becomes the DTFT of $x[n]$. The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:



Relation between Z-Transform and DTFT



Region of convergence

The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.

The Fourier transform does not converge for all sequences — the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the **region of convergence (ROC)**.

The Fourier transform of $x[n]$ exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. However, the z-transform of $x[n]$ is just the Fourier transform of the sequence $x[n]r^{-n}$. The z-transform therefore exists (or converges) if

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

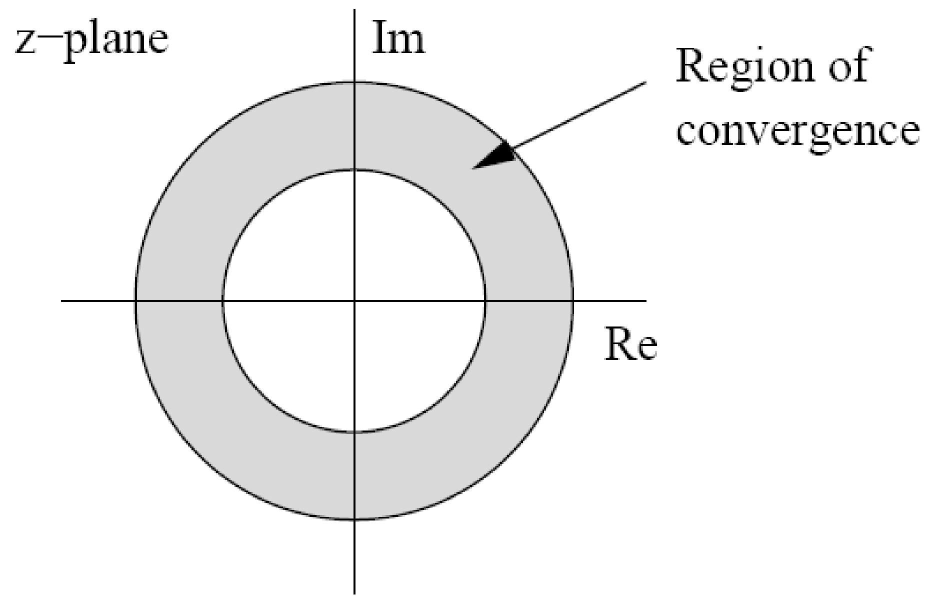
Region of Convergence

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:

Ring of Convergence in the z-plane



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle $|z| = 1$, then the Fourier transform will converge.

Examples of z-transform

- The z-transform of the bilateral sequence $x(n) = 2\delta(n+1) + \delta(n) + 4\delta(n-2)$ is:

$$X(z) = \sum_{n=-\infty}^{\infty} \left[2\delta(n+1) + \delta(n) + 4\delta(n-2)\right] z^{-n} =$$

$$= 2\sum_{n=-\infty}^{\infty} \delta(n+1) z^{-n} + \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} + 4\sum_{n=-\infty}^{\infty} \delta(n-2) z^{-n} =$$

$$= 2z + 1 + 4z^{-2}$$

Geometric series

- The general formula for converging geometric series is:

$$\sum_{n=0}^{k} q^n = \frac{1 - q^{k+1}}{1 - q}$$

- For infinite series the convergence request is: $|q|<1$

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1 - q}.$$

Examples of the Z- transform

- The z transform for the step function $u(n)$

$$X(z)=\sum_{n=-\infty}^{\infty}u(n)z^{-n}=\sum_{n=0}^{\infty}z^{-n}=\sum_{n=0}^{\infty}\left(z^{-1}\right)^{n}=\frac{1}{1-z^{-1}}$$

$$x(n)=\delta(n) \qquad X(z)=\sum_{n=-\infty}^{+\infty}\delta(n)z^{-n}=1$$

Causal

$$x(n)=\delta(n-k) \quad k>0$$

$$X(z)=\sum_{n=-\infty}^{+\infty}\delta(n-k)z^{-n}=z^{-k}$$

Anticausal

$$x(n)=\delta(n+k) \quad k>0$$

$$X(z)=\sum_{n=-\infty}^{+\infty}\delta(n+k)z^{-n}=z^{k}$$

Examples of the Z- transform

$$x_1(n) = \{\underset{\uparrow}{1},2,5,7,0,1\}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n} = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 1z^{-5}$$

$$x(n) = \{1,2,\underset{\uparrow}{5},7,0,1\}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n} = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$

Finite causal sequence z-transform

- Given the following sequence:
- $x(n) = \alpha^n [u(n) - u(n - N)]$,
- Where N is an integer and α is a real constant.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{N-1} \alpha^n z^{-n} =$$

$$= \sum_{n=0}^{N-1} \left(\alpha z^{-1}\right)^n = \frac{1 - \alpha^N z^{-N}}{1 - \alpha z^{-1}}$$

Example of a finite causal sequence

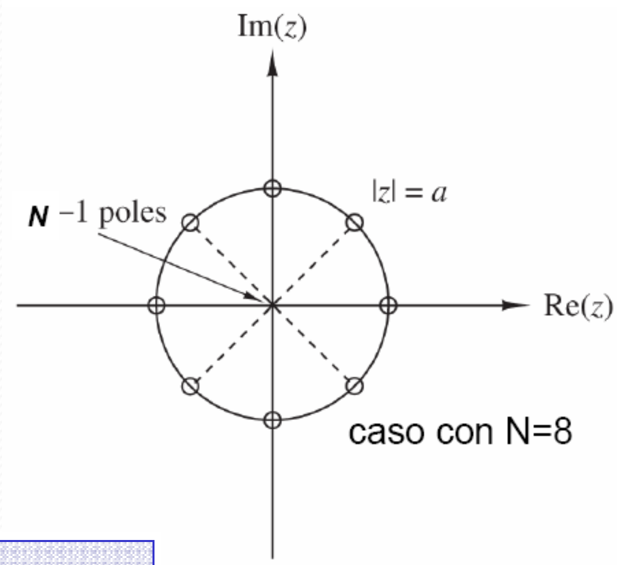
- **Poles** are the roots of the *denominator*
- **Zeros** are the roots of the *numerator*

$$X(z) = \frac{1-\alpha^{N} z^{-N}}{1-\alpha z^{-1}} = z^{-N+1} \frac{z^{N}-\alpha^{N}}{z-\alpha}$$

- The X(z) has a pole of order N – 1 in the origin and N -1 zeros (the z = α root pole at the denominator is compensated by a zero in the same position).

Example of a finite causal sequence

- The polynomial N: $z^N - \alpha^N$ has N zeros uniformly distributed along the circle of radius α.
- The roots of the polynomial are at these complex pulsations.



$$z = \alpha e^{j\frac{2\pi}{N}k} \quad k = 0,\ldots,N-1$$

$$X(z) = \frac{1-\alpha^N z^{-N}}{1-\alpha z^{-1}} = z^{-N+1}\frac{z^N - \alpha^N}{z-\alpha}$$

$$z = \alpha e^{j\frac{2\pi}{N}k} \quad k = 1,\ldots,N-1$$

Example of an infinite causal sequence

- Find the z-transform of the following sequence:

$$\boxed{x(n) = \left(\frac{1}{2}\right)^n u(n)}$$

$$X(z) = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

- It is a geometric series of ratio: $\frac{1}{2} z^{-1}$ converging at:

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

Rational Z transform

- For most common cases *X(z) is the ratio between two polynomials:*

$$X(z) = N(z)/D(z)$$

- Where *N(z) e D(z) are two polynomials of variable* z^{-1}, *of degree* p_n *and* p_d *respectively.*
- The extended notation of the two polynomials are N(z) and D(z):

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_{p_n} z^{-p_n}}{a_0 + a_1 z^{-1} + \ldots + a_{p_d} z^{-p_d}}$$

Rational Z transform

- The z-transform can be expressed through the factoriziation of both numerator and denominator:

$$X(z) = \frac{b_0}{a_0}\left(z^{p_d - p_n}\right)\frac{\prod_{i=1}^{p_n}(z - c_i)}{\prod_{i=1}^{p_d}(z - d_i)}$$

LTI systems analysis by the z-transform

- Time discrete LTI systems can be described as finite differences linear equations with constant coefficients.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \ldots - a_M y(n-M) + \\ + b_0 x(n) + b_1 x(n-1) + \ldots + b_N x(n-N)$$

- Applying the DTFT to each term:

$$Y\left(e^{j\omega}\right) = -a_1 Y\left(e^{j\omega}\right) e^{-j\omega} - \ldots - a_M Y\left(e^{j\omega}\right) e^{-j\omega M} + \\ + b_0 X\left(e^{j\omega}\right) + b_1 X\left(e^{j\omega}\right) e^{-j\omega} + \ldots + b_N X\left(e^{j\omega}\right) e^{-j\omega N}$$

- Remember: $\mathrm{F}\left(f(x-x_0)\right) = e^{-j2\pi u x_0} F(u)$

LTI system analysis by the z-transform

- Gathering all the terms in *$Y(e^{j\omega})$ and $X(e^{j\omega})$ we obtain*

$$Y\left(e^{j\omega}\right)\left(1+a_1e^{-j\omega}+\ldots+a_Me^{-j\omega M}\right)=X\left(e^{j\omega}\right)\left(b_0+b_1e^{-j\omega}+\ldots+b_Ne^{-j\omega N}\right)$$

- *Thanks to the convolution theorem:*

$$y(n)=x(n)*h(n) \quad \Leftrightarrow \quad Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

- The frequency response *$H(e^{j\omega})$ for an LTI system defined by finite differences equations can be defined as*:

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{\left(b_0+b_1e^{-j\omega}+\ldots+b_Ne^{-j\omega N}\right)}{\left(1+a_1e^{-j\omega}+\ldots+a_Me^{-j\omega M}\right)}$$

LTI system analysis by the z-transform

- Thanks to the relation between the DTFT and the z-transform :

$$\sum_{n=-\infty}^{\infty} x(n) z^{-n} = \underbrace{\sum_{n=-\infty}^{\infty} \left(x(n)\rho^{-n}\right) e^{-j\omega n}}_{\text{DTFT } (x(n)\rho^{-n})}$$

- Through the application of the convolution theorem the response of the complex frequency system Z is:

$$Y(z) = Z\left[x(n) * h(n)\right] = X(z)H(z)$$

LTI system analysis by the z-transform

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{\left(b_0+b_1e^{-j\omega}+\ldots+b_Ne^{-j\omega N}\right)}{\left(1+a_1e^{-j\omega}+\ldots+a_Me^{-j\omega M}\right)}$$

- We know that the transfer function *$H(e^{j\omega})$ is a rational funcion*, so that the transfer function *$H(z)$ is also a rational funcion* like: *$N(z)/D(z)$*

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\left(b_0+b_1z^{-1}+\ldots+b_Nz^{-N}\right)}{\left(1+a_1z^{-1}+\ldots+a_Mz^{-M}\right)}$$

- That corresponds to the finite difference equation.

$$y(n)=-a_1y(n-1)-a_2y(n-2)-\ldots-a_My(n-M)+$$
$$+b_0x(n)+b_1x(n-1)+\ldots+b_Nx(n-N)$$

LTI systems

- In the set of LTI system, described throught the difference equations we can define two kinds of systems.
- FIR, Finite Impulse Response filters: non recursive systems where the output y(n) has a dipendence just from the input signal *x(n):*

$$y(n) = \sum_{k=0}^{N} b_k x(n-k)$$

LTI systems

- IIR, Infinite Impulse Response Filters: systems where the outputs **y(n)** depend both from inputs and from outputs themselves :

$$y(n) = \sum_{k=0}^{N} b_k x(n-k) - \sum_{j=1}^{M} a_j y(n-j)$$

- A sub-set of this kind of systems concern only recursive filters, i.e. where the output depends just from the actual input and from previous outputs.

$$y(n) = x(n) - \sum_{j=1}^{M} a_j y(n-j)$$

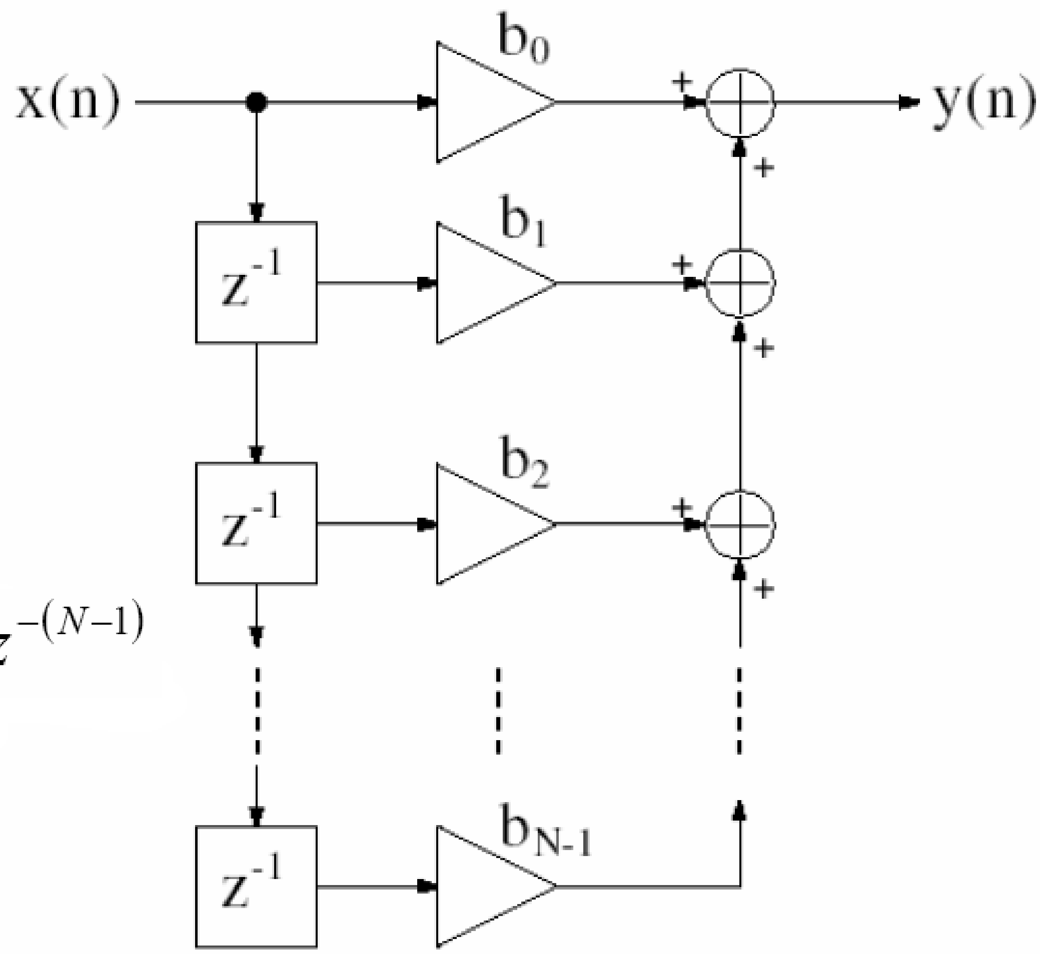
FIR and their z-transform

- A causal LTI non recursive FIR filter

$$y(n)=\sum_{k=0}^{N-1} b_k x(n-k)$$

- It can be described by the equation:

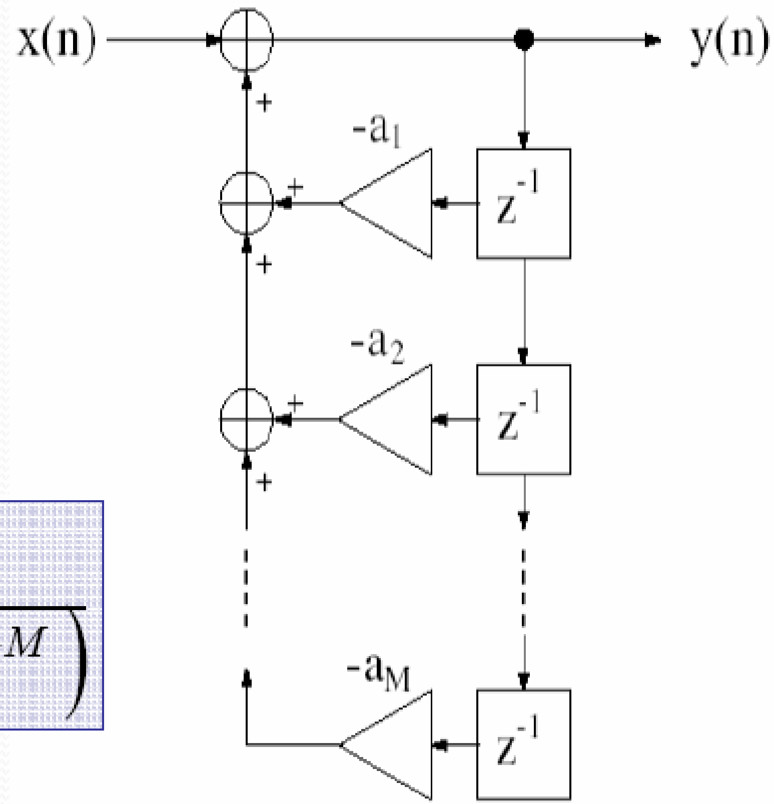
$$H(z)=b_0+b_1 z^{-1}+\ldots+b_{N-1} z^{-(N-1)}$$



Pure recursive IIR filter

- An LTI causal system made of a pure IIR is a filter like:

$$y(n) = x(n) - \sum_{j=1}^{M} a_j y(n-j)$$

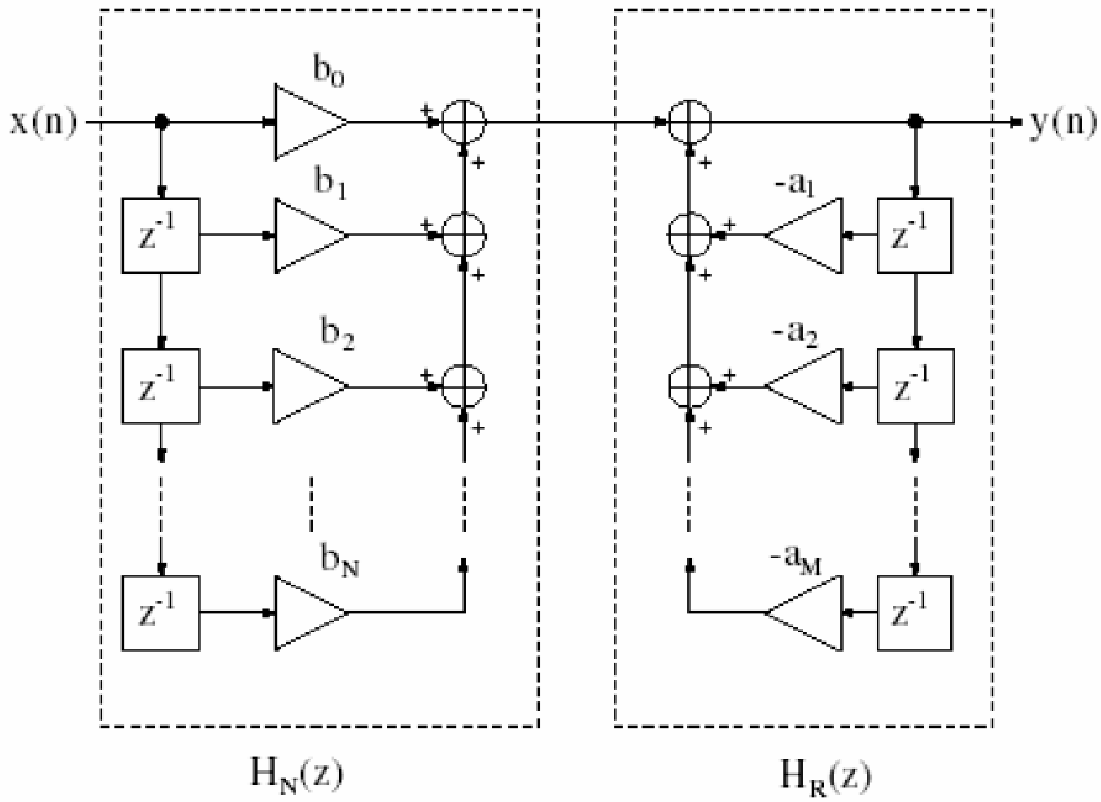


- It can be described using the following equation.

$$H(z) = \frac{1}{\left(1 + a_1 z^{-1} + \ldots + a_M z^{-M}\right)}$$

LTI general case for a LTI system

$$H(z) = \frac{\left(b_0 + b_1 z^{-1} + \ldots + b_N z^{-N}\right)}{\left(1 + a_1 z^{-1} + \ldots + a_M z^{-M}\right)}$$



LTI and z-transform

- The transfer function H(z) can be thought in terms of the roots of the polynomials of the numerator and

$$H(z) = \frac{N(z)}{D(z)} = Kz^{M-N} \frac{(z-c_1)(z-c_2)\ldots(z-c_N)}{(z-d_1)(z-d_2)\ldots(z-d_M)}$$

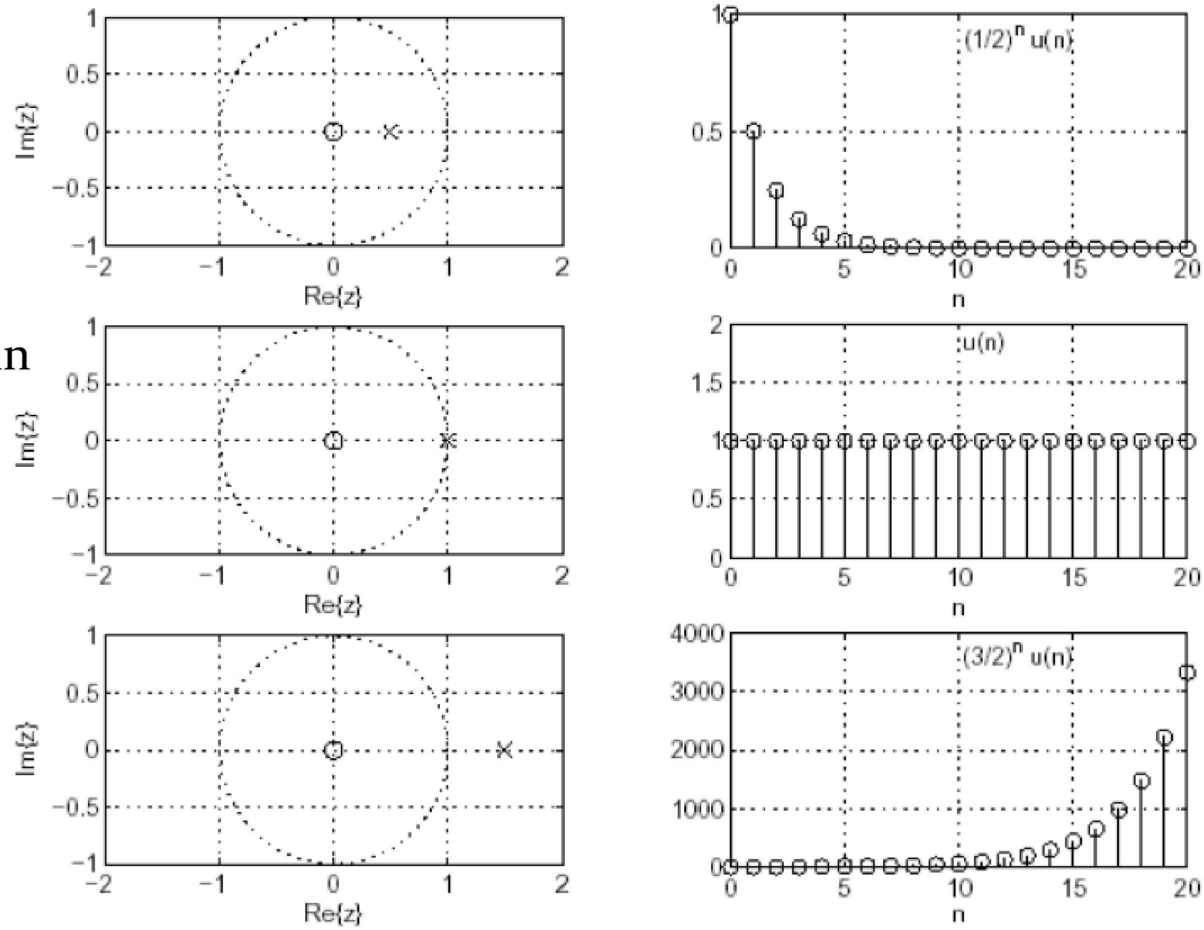
- The output of a LTI system is described and analyzed by its zeros and poles.
- For a causal system the number of zeros cannot be larger then the number of poles .

$x(n) = \alpha^n u(n)$

$$X(z) = \frac{1}{1-\alpha z^{-1}} = z\frac{1}{z-\alpha}$$

$\alpha > 0$

1 pole in α
1 zero in the origin

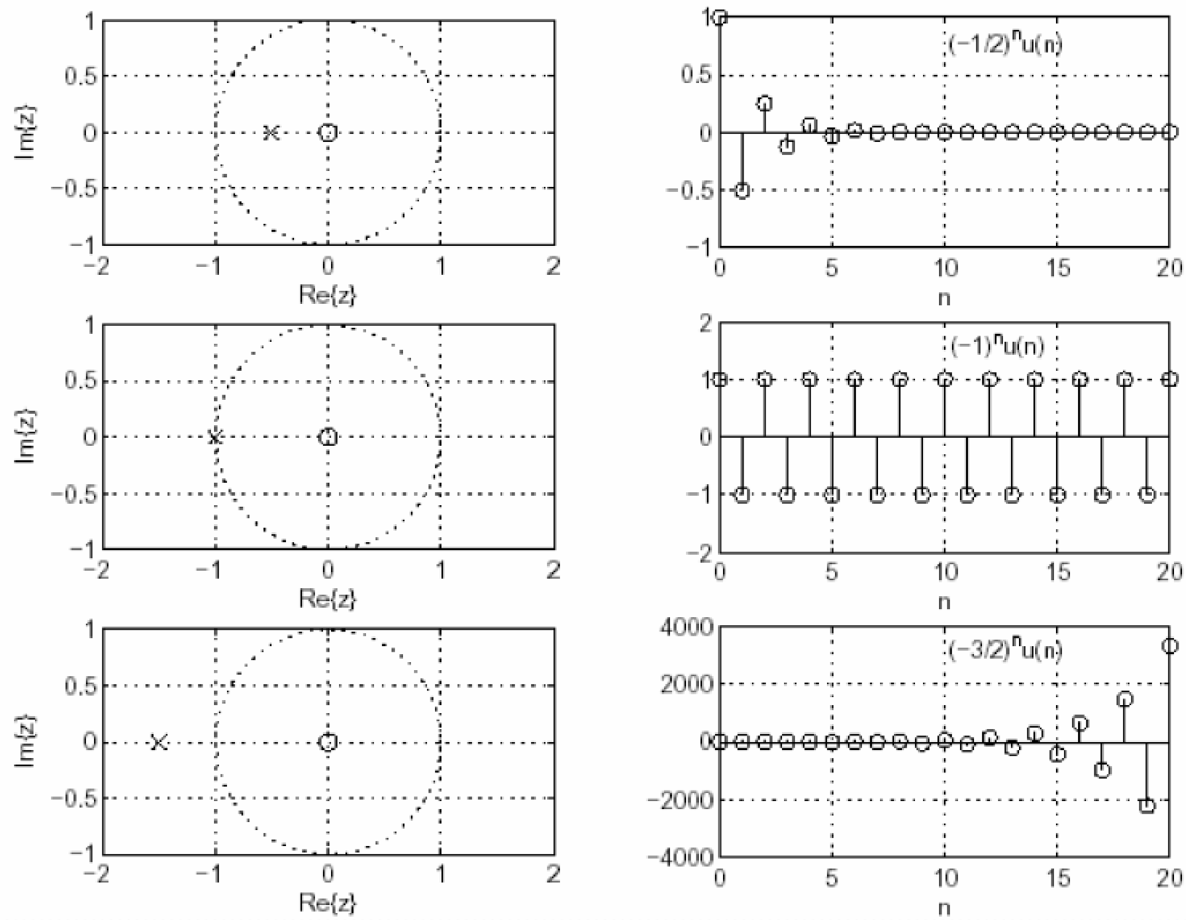


$x(n) = \alpha^n u(n)$

$$X(z) = \frac{1}{1-\alpha z^{-1}} = z\,\frac{1}{z-\alpha}$$

$\alpha < 0$

1 polo in α
1 zero in 0

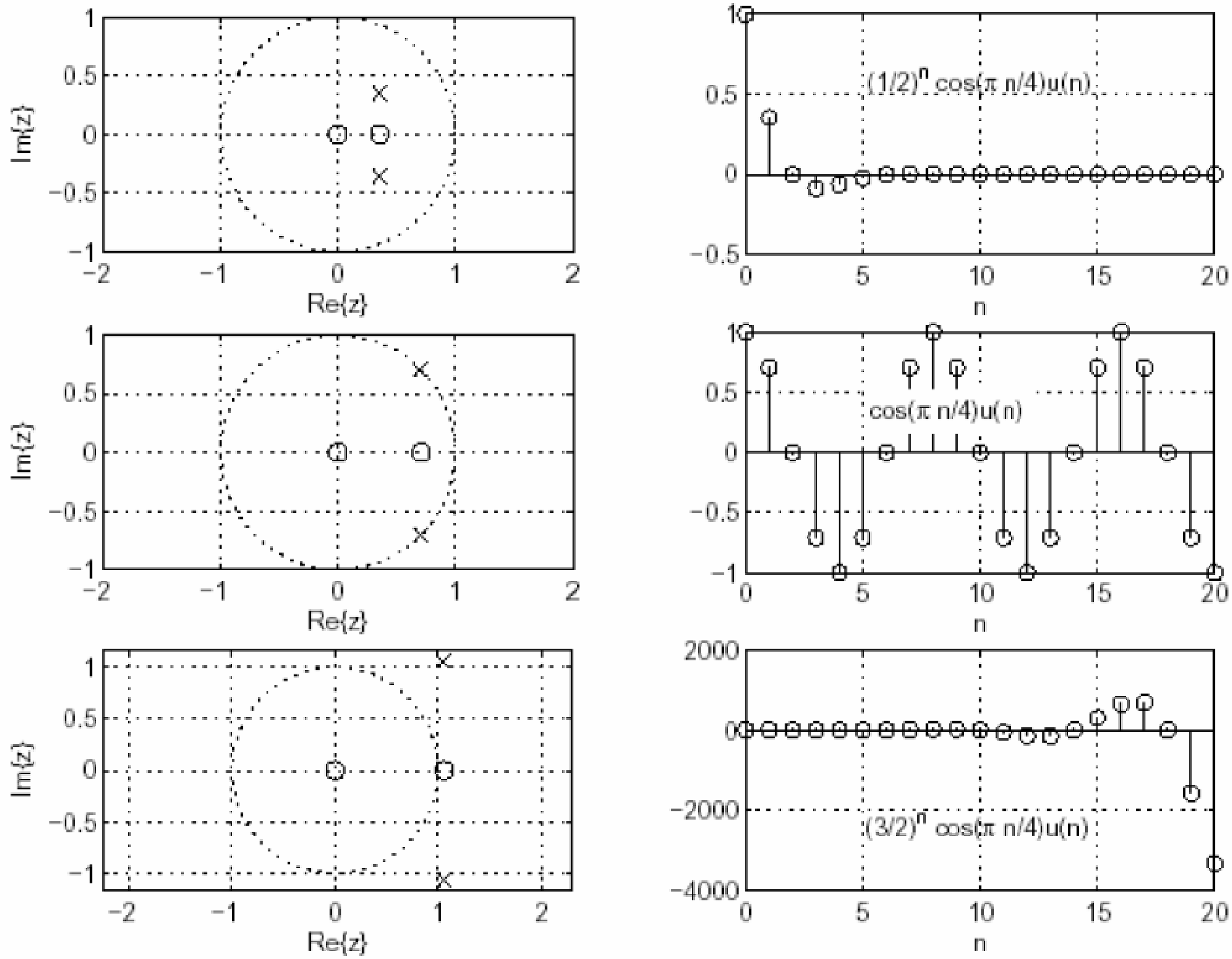


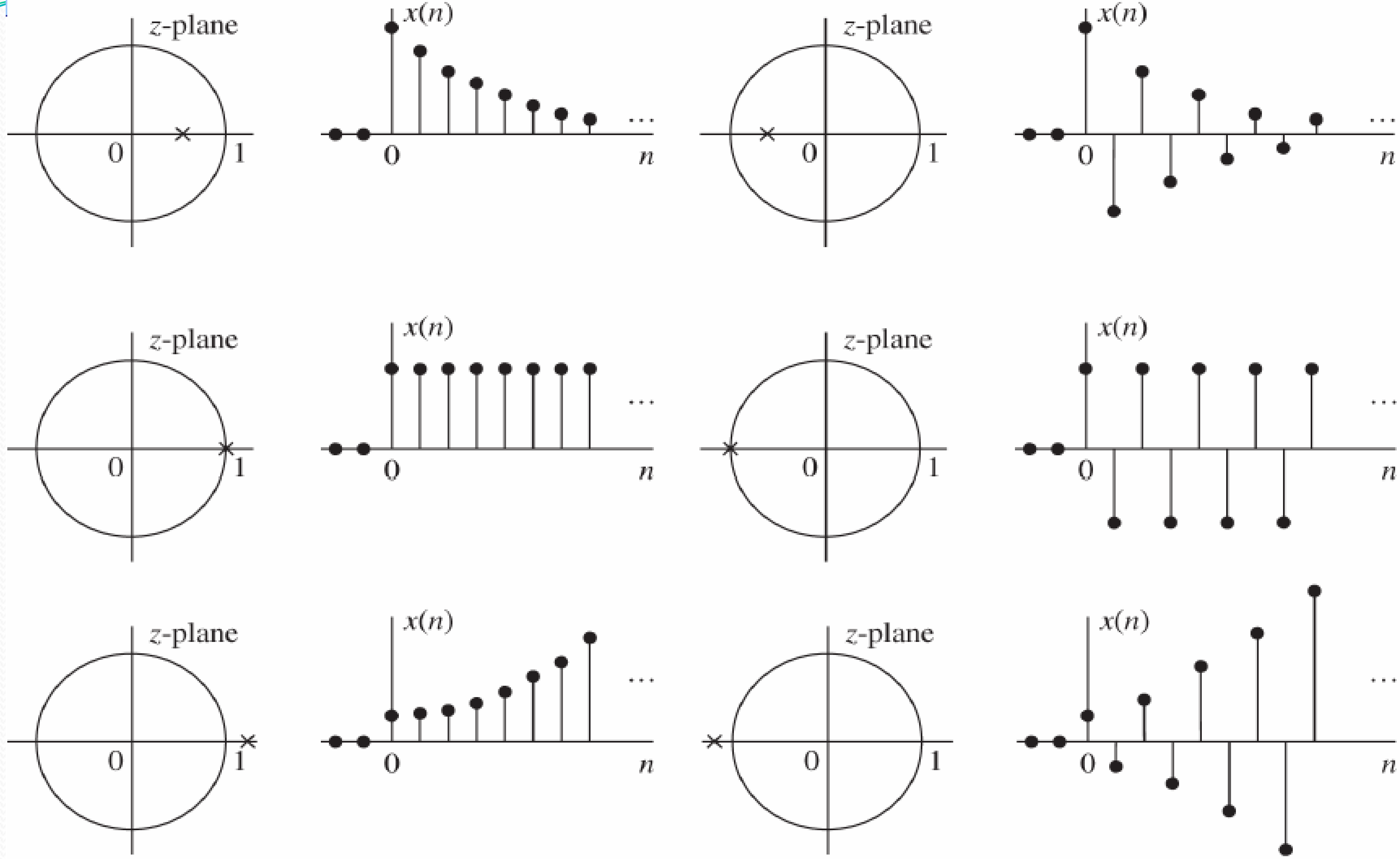
Complex and conjugate poles

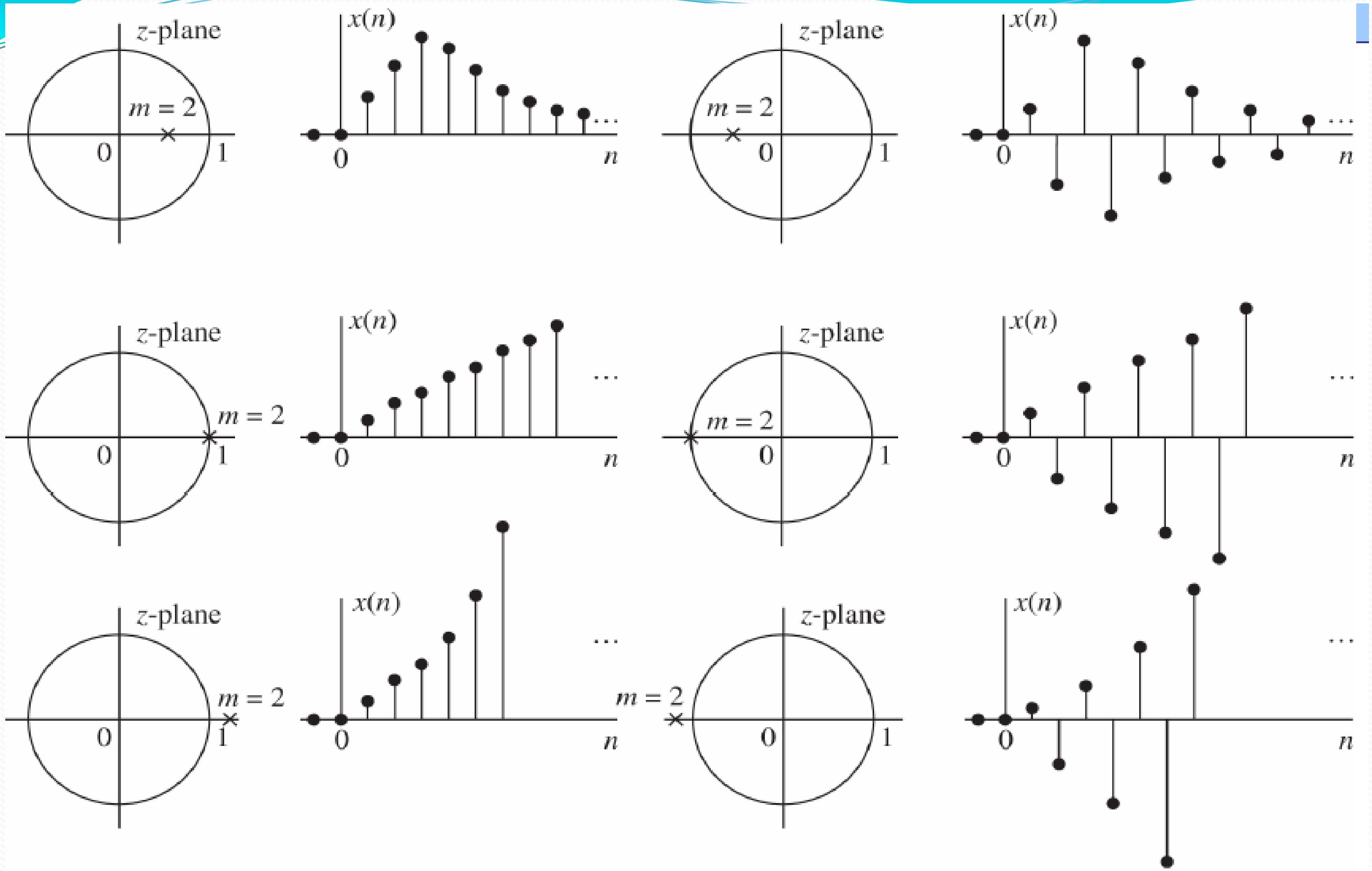
$\alpha > 0$

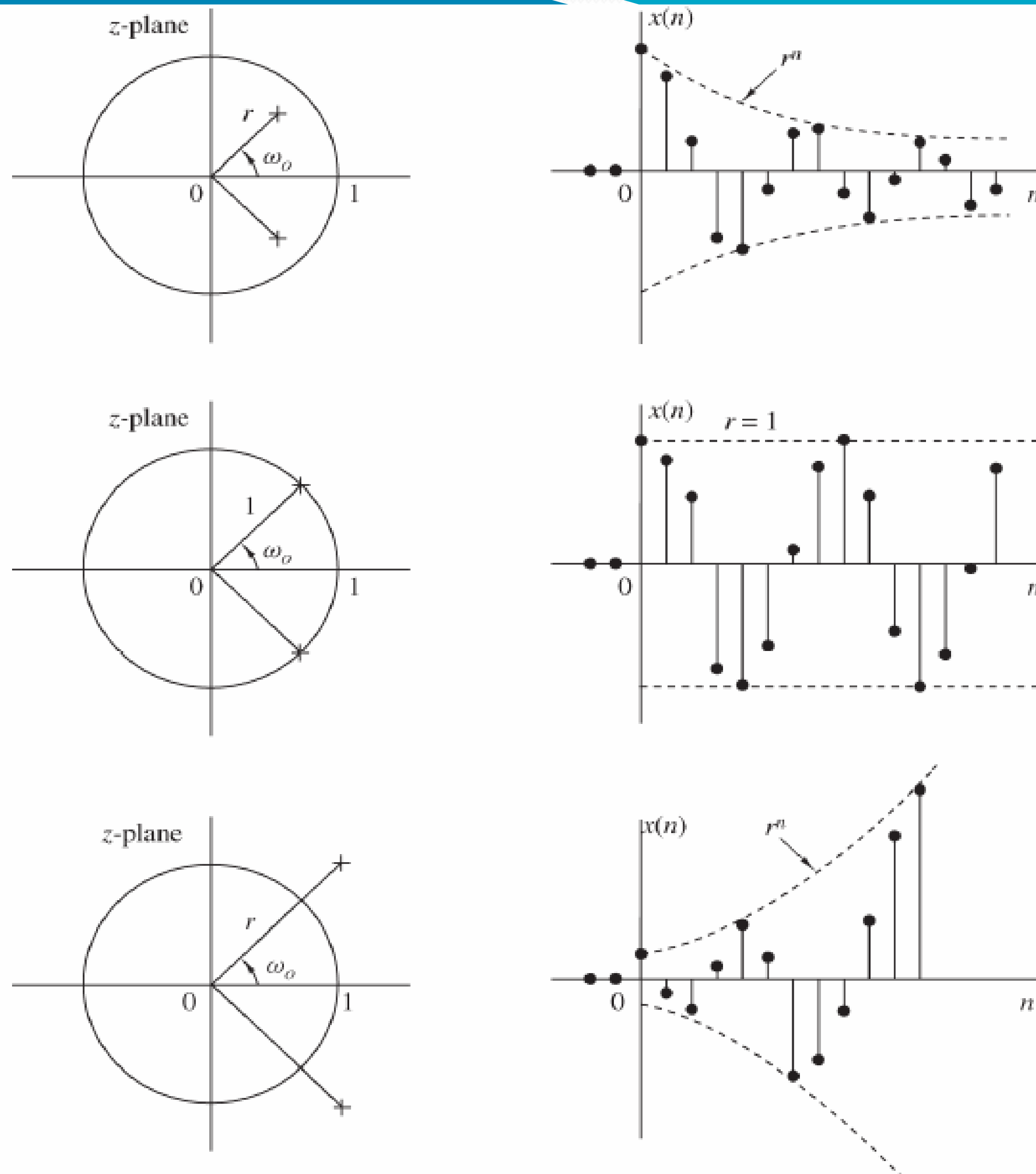
Two compex
and conjugate
poles
and two zeros

e




z-plane
x(n)
0
1
n
...


z-plane
m = 2
0
1
x(n)
0
n
z-plane
m = 2
0
1
x(n)
0
n
z-plane
m = 2
0
1
x(n)
0
n
z-plane
m = 2
0
1
x(n)
0
n
z-plane
m = 2
0
1
x(n)
0
n
z-plane
m = 2
0
1
x(n)
0
n



z-plane
r
ω_o
0
1
x(n)
r^n
0
n
z-plane
1
ω_o
0
1
x(n)
r = 1
0
n
z-plane
r
ω_o
0
1
x(n)
r^n
0
n

FIR

- FIR :

$$\{\underset{\Uparrow}{1},2,0,4\} \quad h(n)=\sum_{j=0}^{N-1} b_j \delta(n-j)$$

- From a generic input x(n), we obtain the output y(n):

$$y(n)=\sum_{k=-\infty}^{+\infty} h(k)x(n-k)=\sum_{k=0}^{3} h(k)x(n-k)$$

$$\boxed{y(n)=x(n)+2x(n-1)+4x(n-3)}$$

- Once h(n) is known the z-transform becomes:

$$\boxed{H(z)=1+2z^{-1}+4z^{-3}}$$

Exercise

- Find the output y(n) of a system with:

$$H(z) = 1 - z^{-2} + 2z^{-3}$$

- When the input is x(n)= u(n);

$$h(n) = \delta(n) - \delta(n-2) + 2\delta(n-3)$$

$$y(n) = x(n) - x(n-2) + 2x(n-3)$$

System stability

- The stability for a LTI system requires that the modulus of impulse response *h(n) is summable*

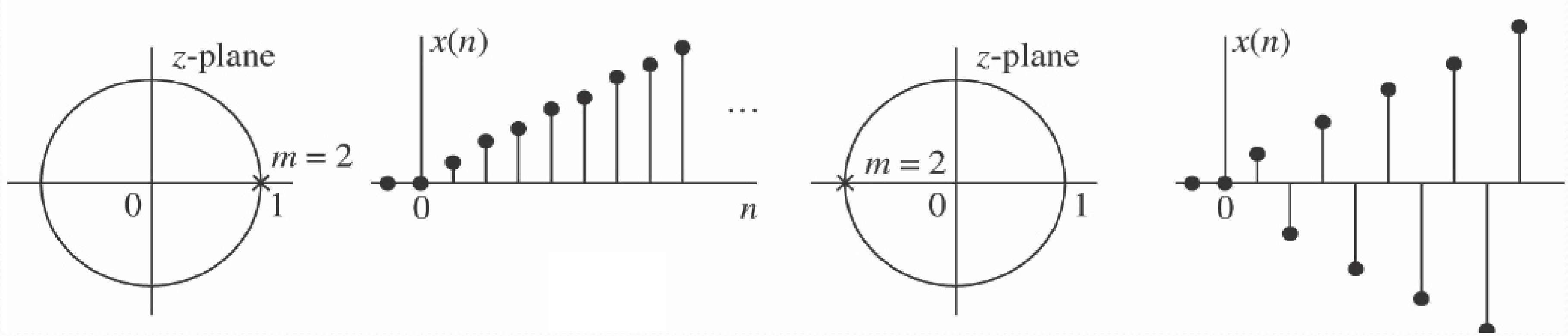
$$\sum_{n=-\infty}^{\infty} \left| h(n) \right| < \infty$$

- If the system is causal the stability condition becomes: in the *z domain the transfer function H(z) has all the poles inside the unitary circle of the z-plane.*

$$H(z) = \frac{z}{z-3}$$

Stability

- Multiple poles on the unitary circle induce a polynomial grow in the impulse response.



- Has a pole of order 2 in z=1
 the impulse response is
 h(n)=n u(n): a slope function (it can be interpreted as the convolution of a step with itself).

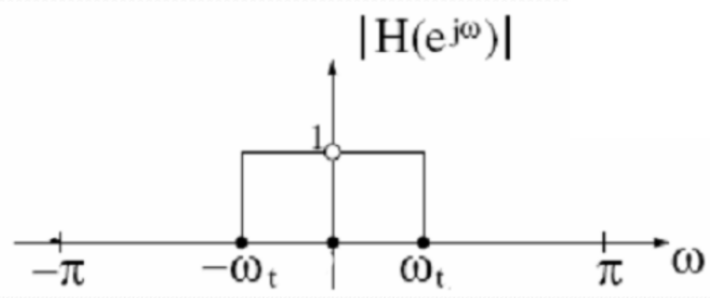
$$H(z)=\frac{1}{z(z-1)^2}$$

Filter design by the positioning of poles and zeros

$$H(z)=\frac{N(z)}{D(z)}=Kz^{M-N}\frac{(z-c_1)(z-c_2)\ldots(z-c_N)}{(z-d_1)(z-d_2)\ldots(z-d_M)}$$

- The poles must be placed close to the unitary circle at the complex pulsations for the frequencies of the input signal *x(n)* that must be emphasised.
- Zeros must be placed closed to the unitary circle at the pulsations of the input signal *x(n) that must be attenuated.*

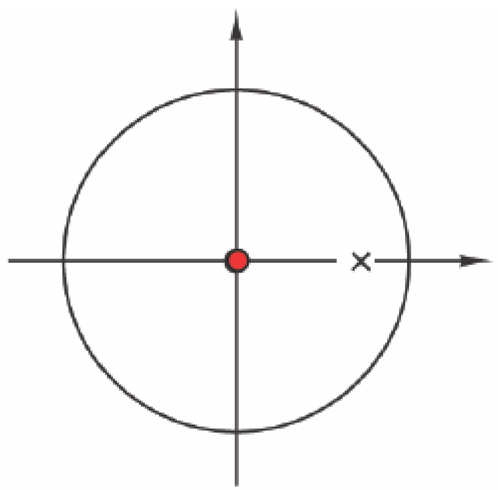
Ideal low pass filter

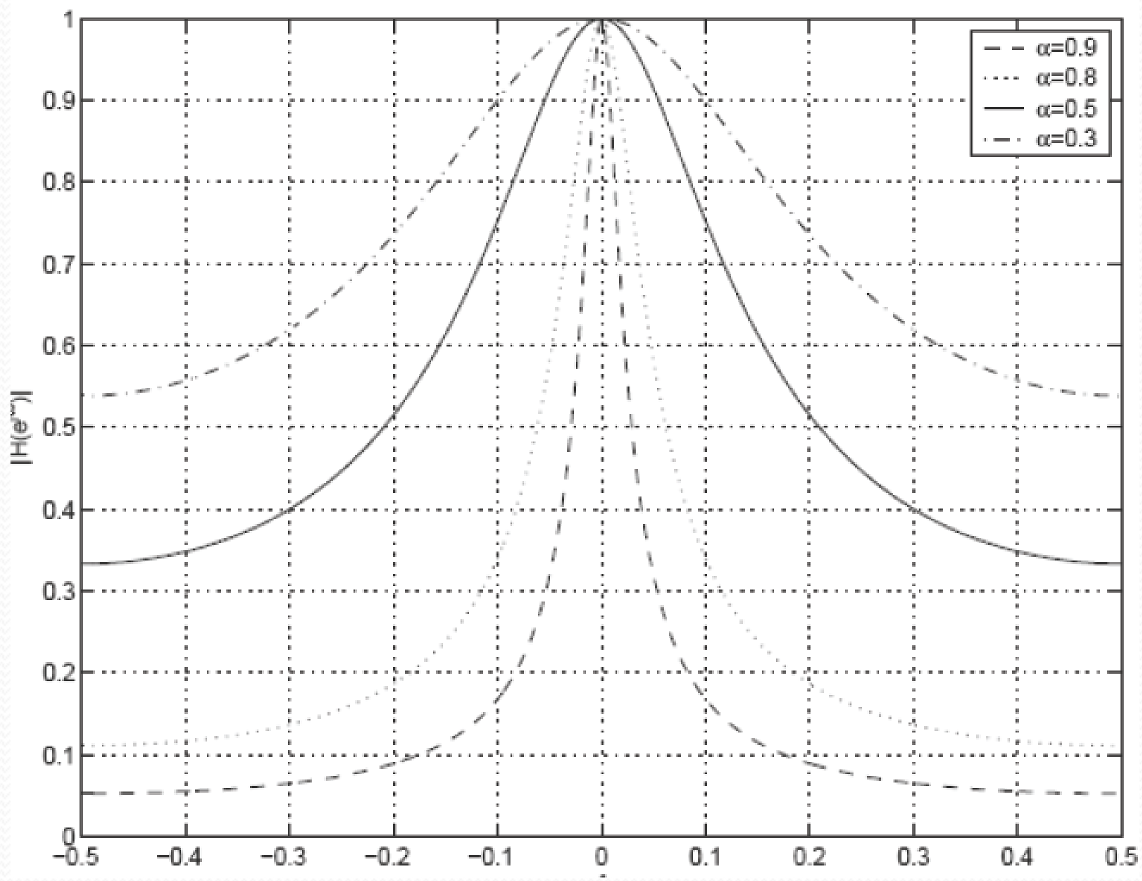


- Filter poles must be placed at pulsations of the pass band $H\left(e^{j\omega}\right):|\omega|\in[0,\omega_t]$
- Zeros must be placed on the unitary circle $|z|=1$, *at the complementary pulsations* $|\omega|\in[\omega_t,\pi]$

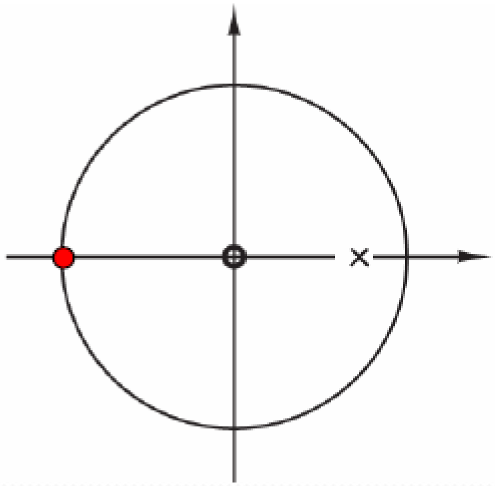
Low pass filter 1 pole 1 zero

$$H_1(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = z\frac{(1-\alpha)}{z-\alpha}$$

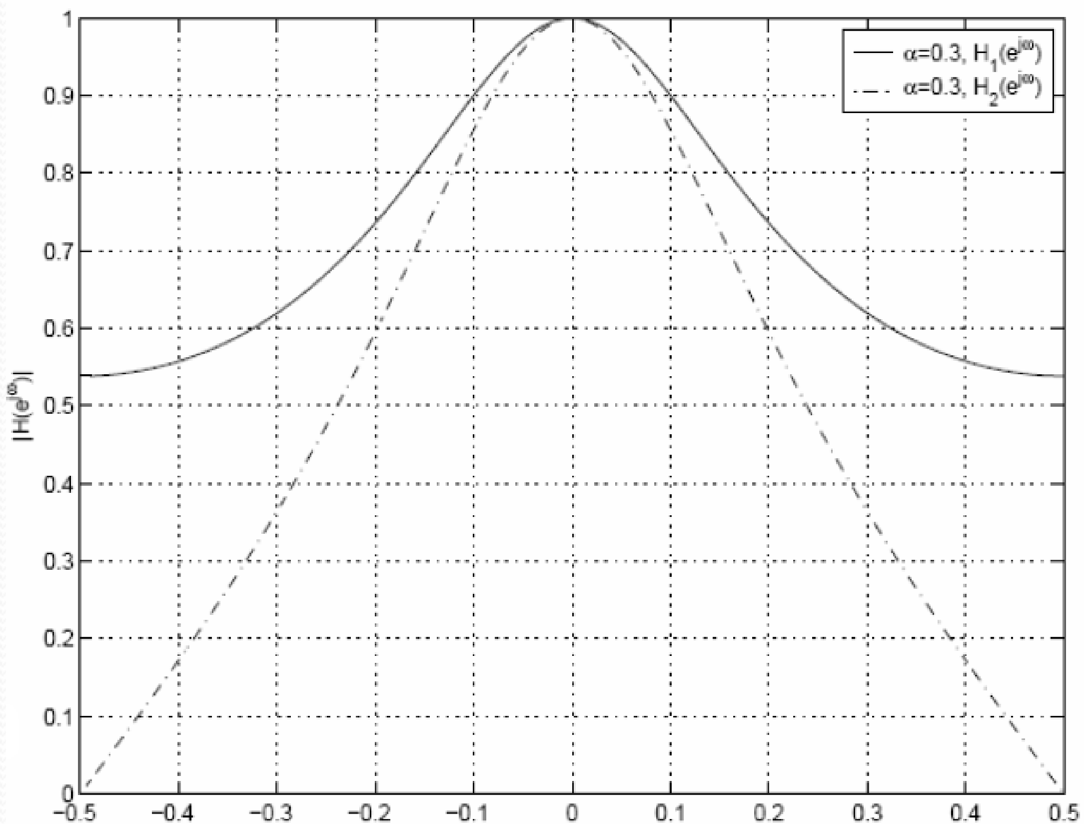




Low pass filter 1 pole 1 zero



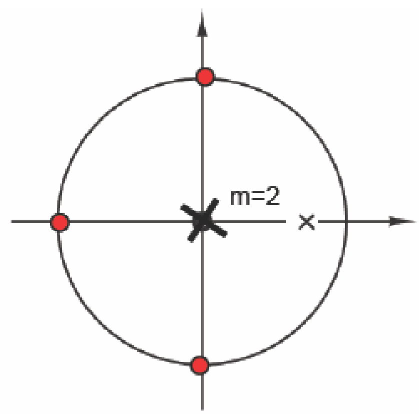
$$H_2(z) = \frac{1-\alpha}{2}\frac{\left(1+z^{-1}\right)}{\left(1-\alpha z^{-1}\right)} = \frac{1-\alpha}{2}\frac{(z+1)}{(z-\alpha)}$$



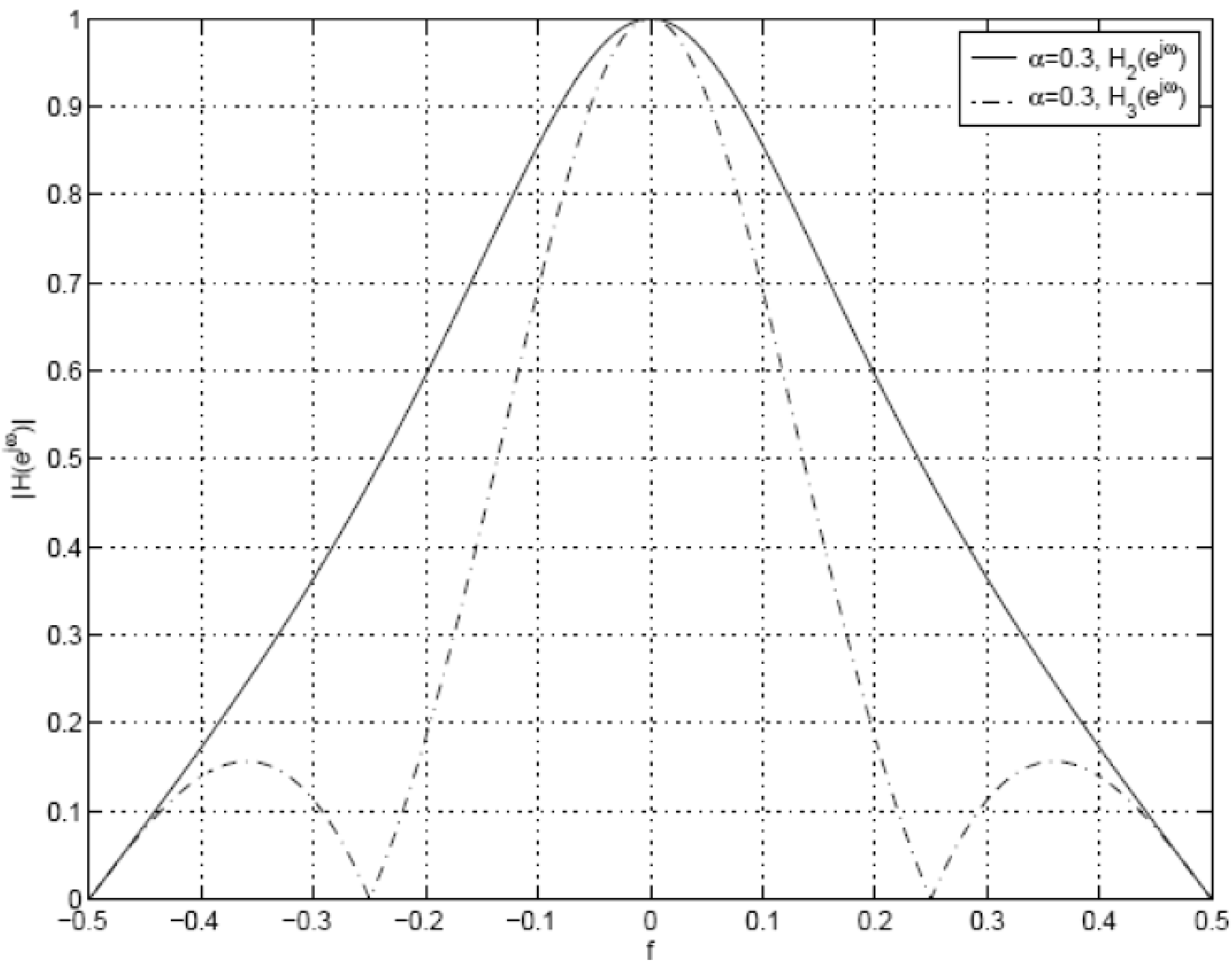
Low pass filter with 3 zeros and 1 pole

- It is possible to emphasize the attenuation of the low pass filter at the high frequencies inserting further couples of complex and conjugate zeros (for the physical realizability of a filter).

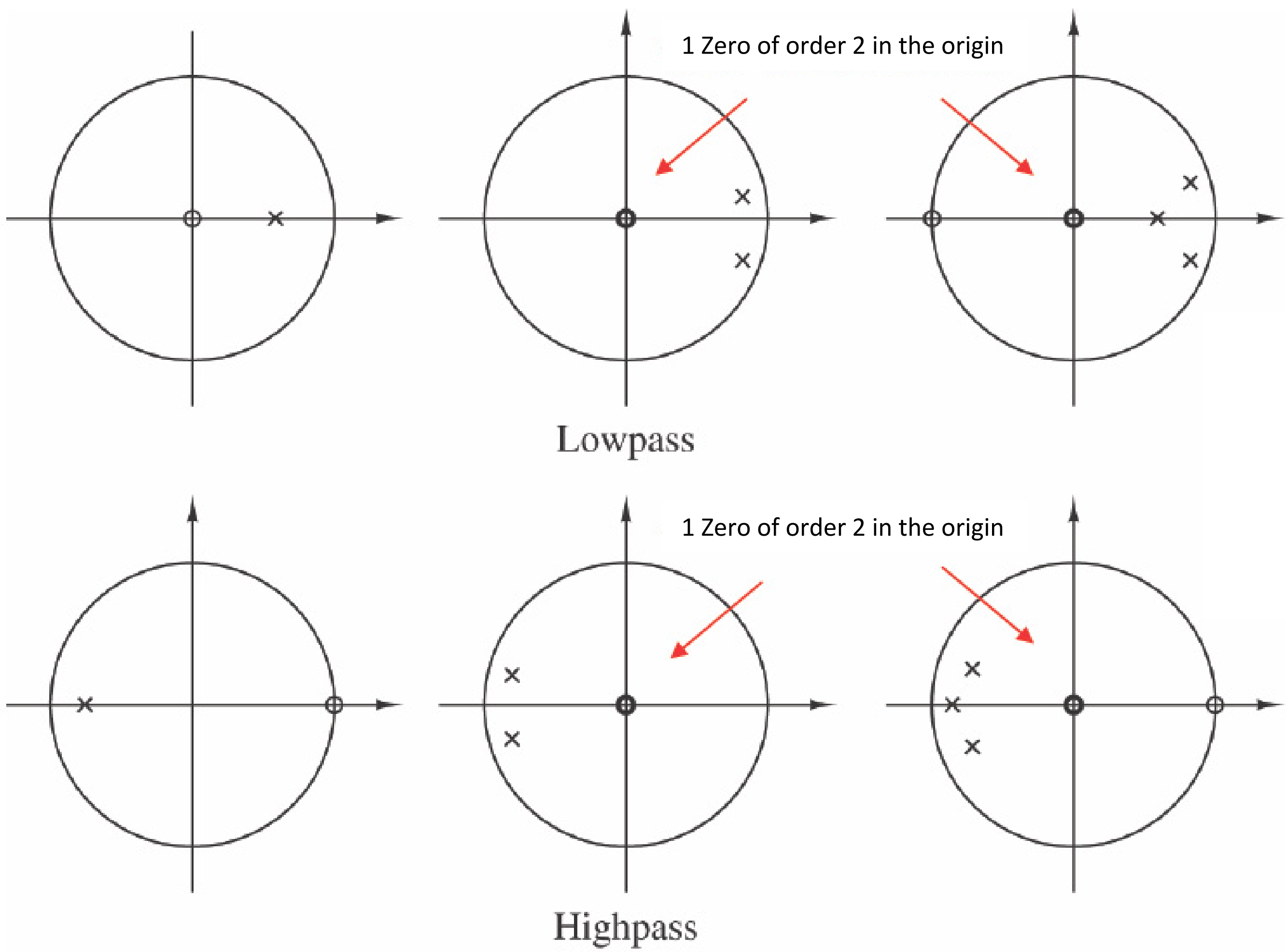
$$H_3(z) = \frac{1-\alpha}{4}\frac{\left(1+z^{-1}\right)}{1-\alpha z^{-1}}\left(1-\beta z^{-1}\right)\left(1-\beta^* z^{-1}\right)$$



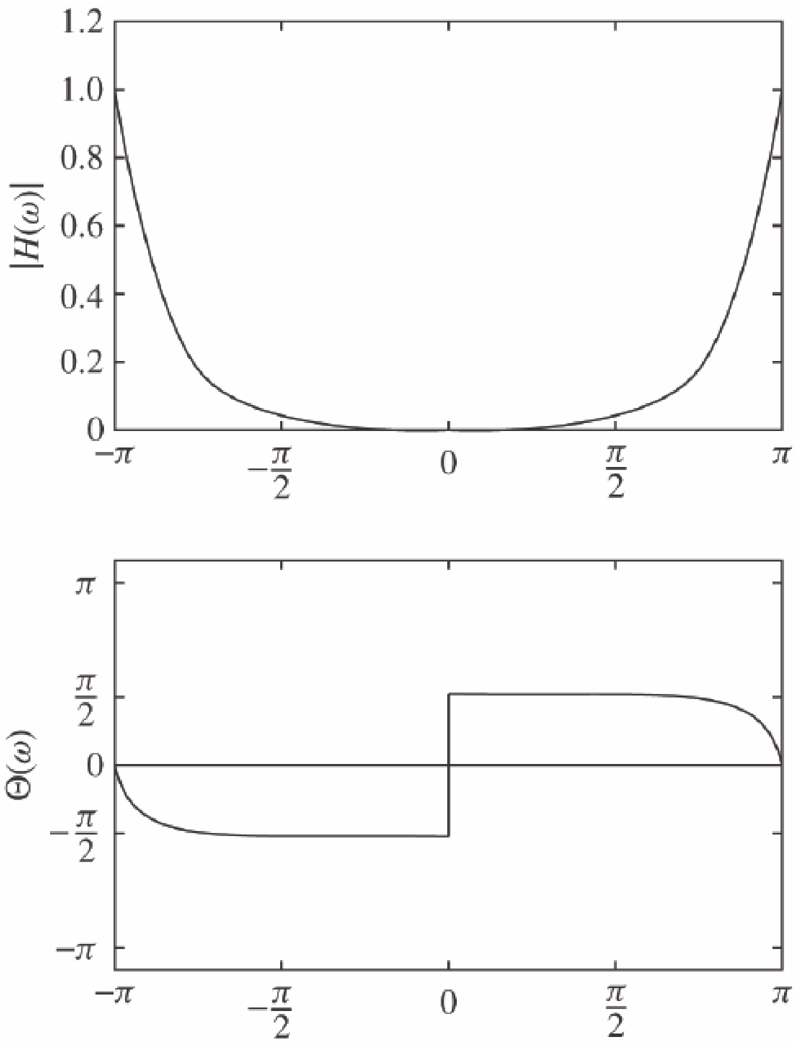
Low pass filter with 3 zeros



Low pass and high pass bands



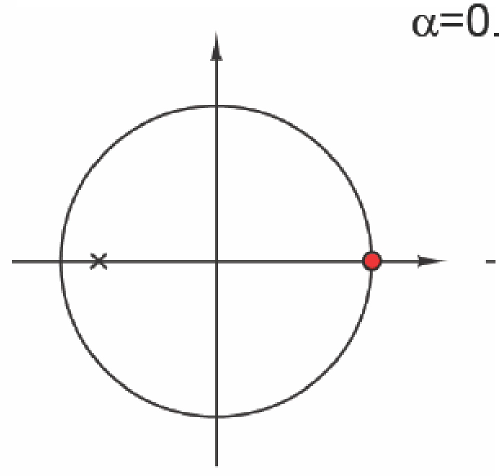
High pass with one pole and one zero



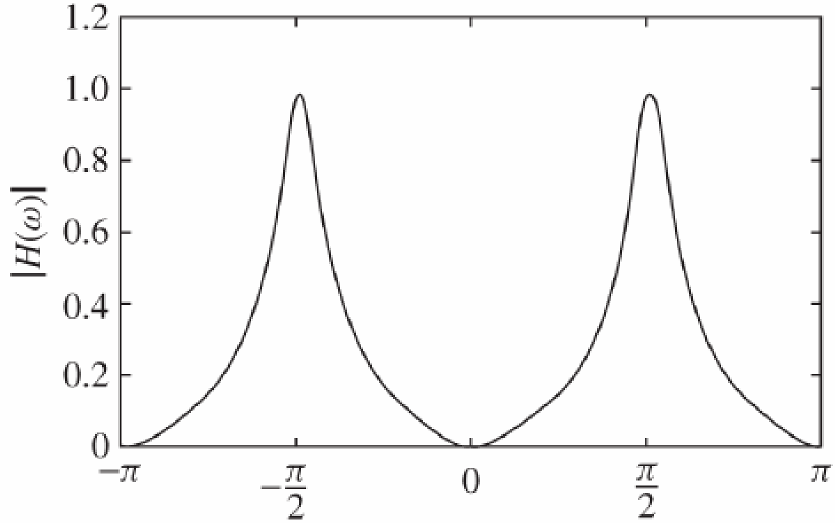
$$H_2(z) = \frac{1-\alpha}{2}\frac{\left(1-z^{-1}\right)}{\left(1+\alpha z^{-1}\right)} = $$

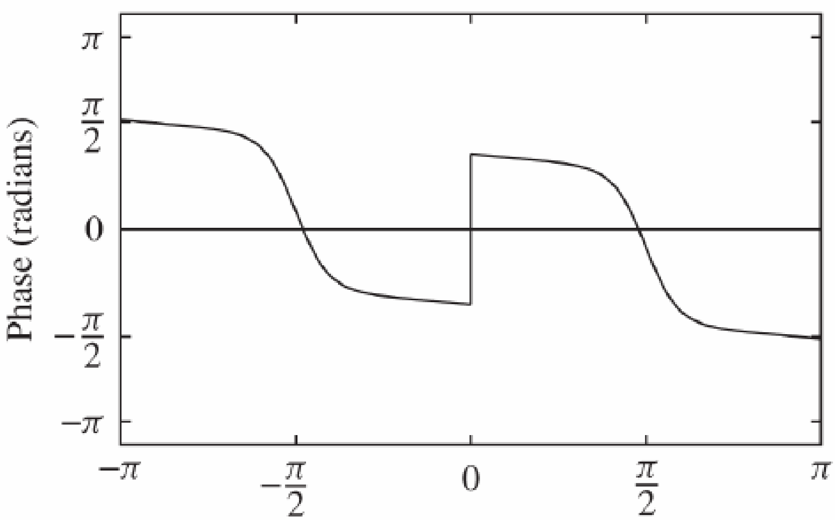
$$= \frac{1-\alpha}{2}\frac{(z-1)}{(z+\alpha)}$$

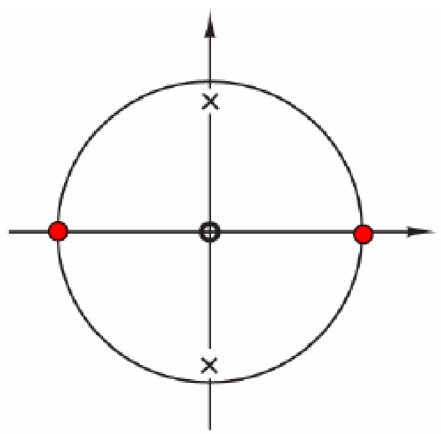
α=0.9



Band pass filter







Amplitude and phase definition

- Two samples sequence: one zero system.

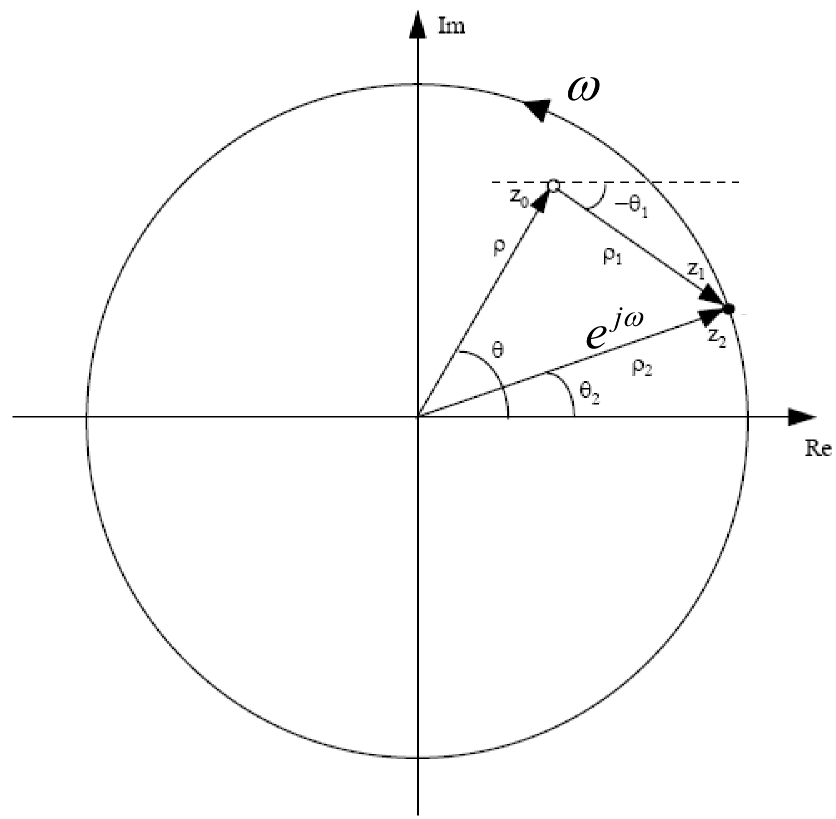
$$A(z) = 1 - z_0 z^{-1} = \frac{(z - z_0)}{z}$$

- Amplitude characteristic

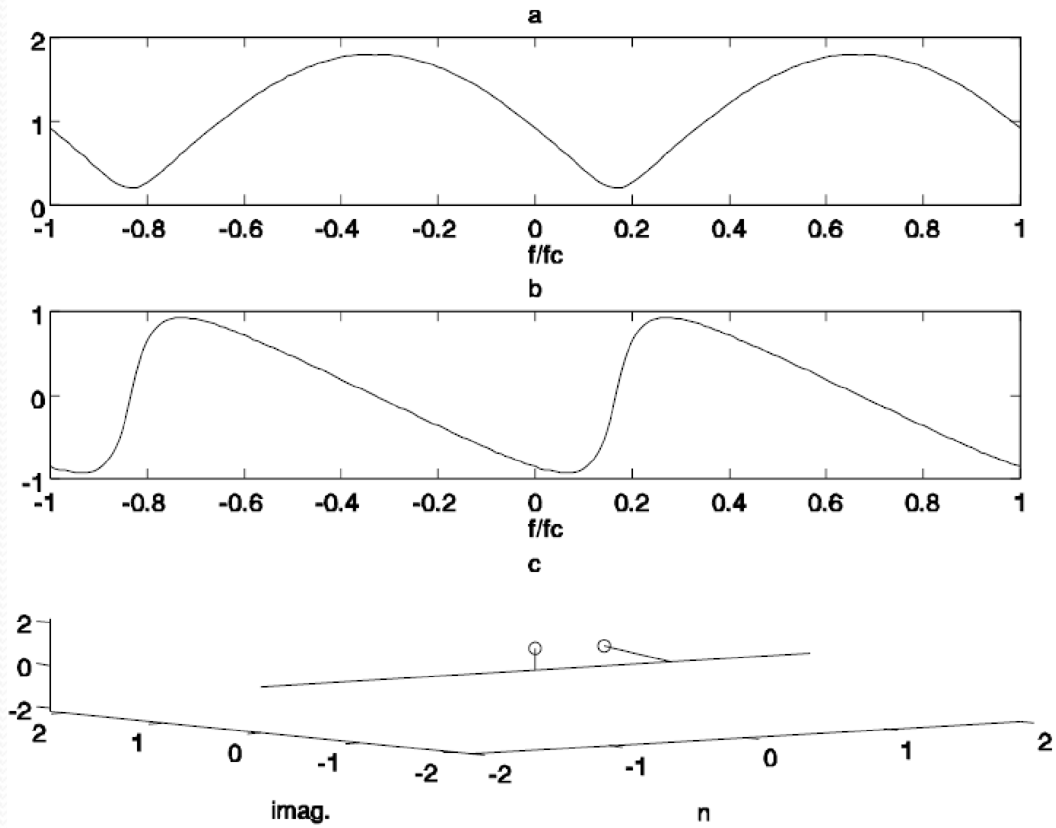
$$\left|A(z)\right|_{z=e^{j\omega}} = \frac{\left|z - z_0\right|}{\left|z\right|} = \left|z - z_0\right|$$

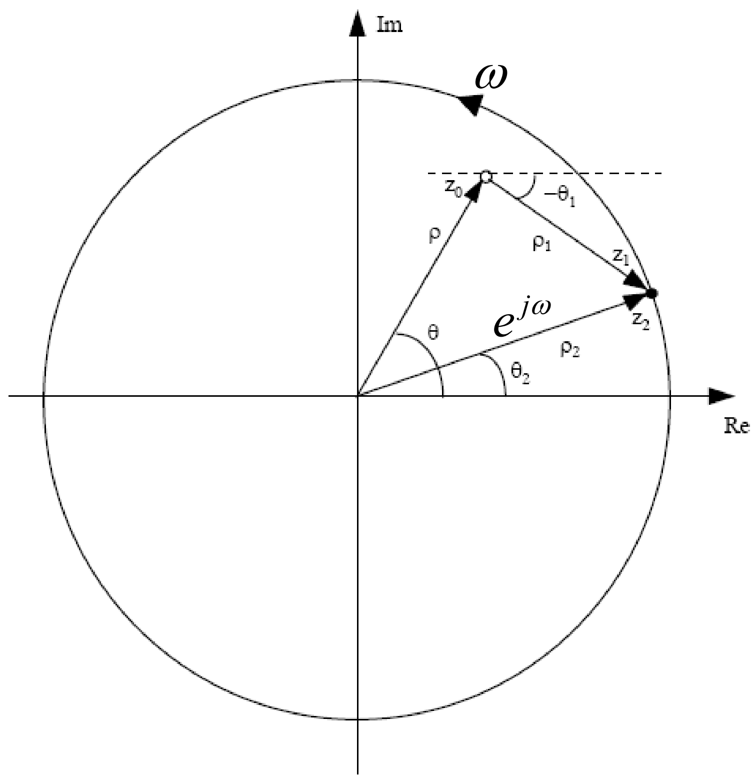
- Phase characteristic

$$\measuredangle A(z)\Big|_{z=e^{j\omega}} = \measuredangle(z - z_0) - \measuredangle(z)$$



Amplitude and phase characteristic





Zeros for maximum and minimum phase

- If zeros are on the unit circle they will delete true sinusoids.
- If they are outside they will delete generalized growing sinusoids.
- While, if they are inside they will delete generalized decreasing sinusoids.
- The zeros inside the unit circle are minimum phase zeros while the ones outside are called maximum phase zeros.

Analysis for one zero

$$A(z) = 1 - z^{-1}$$

- The phase behavior in the origin presents a discontinuity:

$$\measuredangle\left(A(z)\big|_{z=e^{j\omega}}\right) = \measuredangle\left(1-e^{-j\omega}\right) =$$

$$= \measuredangle\left(e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)\cdot\frac{2j}{2j}\right) = \measuredangle\left(2je^{-j\omega/2}\sin\frac{\omega}{2}\right) =$$

$$= \measuredangle\left(e^{j\pi/2}e^{-j\omega/2}\sin\frac{\omega}{2}\right) = \begin{cases} \frac{\pi}{2}-\frac{\omega}{2} & \text{for } \sin\frac{\omega}{2}\geq 0 \text{ i.e. } \omega\geq 0 \\ \frac{\pi}{2}-\frac{\omega}{2}-\pi = -\frac{\pi}{2}-\frac{\omega}{2} & \text{for } \sin\frac{\omega}{2}< 0 \text{ i.e. } \omega< 0 \end{cases}$$

- While, if we consider a zero not on the unit circle we will obtain:

$$\measuredangle\left(1-\rho z^{-1}\Big|_{z=e^{j\omega}}\right)$$

- If the zero is inside the unit circle the phase characteristic for the zero frequency is null and continuous.

Reciprocal and conjugate zeros

- When a zero is really close to the unit circle minimal variations of the sequence samples can generate high variations in the phase characteristic.
- It is interesting to notice that two zeros reciprocal and conjugate zeros (i.e. two sequences, each made of 2 samples, whose z-transforms present one zero in z_o or one zero in $1/ z_o^*$) present the same amplitude response (apart from a constant coefficient) while have completely different phase characteristic.

$$\left|A(z)\right|^2 = \left|1 - z_0 z^{-1}\right|^2 = \left|1 - z_0 e^{-j\omega}\right|^2 = 1 + \left|z_0\right|^2 - 2\,\mathrm{Re}\left(z_0 e^{-j\omega}\right)$$

$$\left|A_r(z)\right|^2 = \left|1 - \frac{1}{z_o^*} z^{-1}\right|^2 = \frac{\left|z_o^* - e^{-j\omega}\right|^2}{\left|z_0\right|^2} = \frac{1 + \left|z_0\right|^2 - 2\,\mathrm{Re}\left(z_0 e^{-j\omega}\right)}{\left|z_0\right|^2} = \frac{\left|A(z)\right|^2}{\left|z_0\right|^2}$$

The inverse z-transform

Formally, the inverse z-transform can be performed by evaluating a Cauchy integral. However, for discrete LTI systems simpler methods are often sufficient.

3.1 Inspection method

If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection. For example, one can invert the z-transform

$$X(z) = \left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2},$$

using the z-transform pair

$$a^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad \text{for } |z| > |a|.$$

Inspection method (cont.)

By inspection we recognise that

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Also, if $X(z)$ is a sum of terms then one may be able to do a term-by-term inversion by inspection, yielding $x[n]$ as a sum of terms.

Partial fraction expansion

For any rational function we can obtain a partial fraction expansion, and identify the z-transform of each term. Assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

It is always possible to factor $X(z)$ as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})},$$

where the c_k's and d_k's are the nonzero zeros and poles of $X(z)$.

Partial fraction expansion

- If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}.$$

In this case the coefficients A_k are given by

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}.$$

- If $M \geq N$ and the poles are all first order, then an expansion of the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

can be used, and the B_r's be obtained by long division of the numerator by the denominator. The A_k's can be obtained using the same equation as for $M < N$.

Power series expansion

If the z-transform is given as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \ldots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots,$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of z^{-1}.

Finite length sequence

Example: finite-length sequence

The z-transform

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

By inspection, the corresponding sequence is therefore

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$x[n] = 1\delta[n+2] - \frac{1}{2}\delta[n+1] - 1\delta[n] + \frac{1}{2}\delta[n-1].$$

Power series expansion

Consider the transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore divide to get a power series in powers of z^{-1}:

$$\begin{array}{r l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \cline{2-2}
1 - az^{-1} \big) & 1 \\
 & 1 - az^{-1} \\ \cline{2-2}
 & \quad az^{-1} \\
 & \quad az^{-1} - a^2z^{-2} \\ \cline{2-2}
 & \qquad a^2z^{-2} + \cdots
\end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots .$$

Properties of the z-transform

- Linearity

The linearity property is as follows:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_1}.$$

Time shifting

The time-shifting property is as follows:

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z), \qquad \text{ROC} = R_x.$$

(The ROC may change by the possible addition or deletion of $z=0$ or $z=\infty$.) This is easily shown:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-(m+n_0)}$$
$$= z^{-n_0}\sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-n_0}X(z).$$

Example: shifted exponential sequence

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \frac{1}{4}.$$

From the ROC, this is a right-sided sequence. Rewriting,

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \frac{1}{4}.$$

The term in brackets corresponds to an exponential sequence $(1/4)^n u[n]$. The factor z^{-1} shifts this sequence one sample to the right. The inverse z-transform is therefore

$$x[n] = (1/4)^{n-1} u[n-1].$$

Note that this result could also have been easily obtained using a partial fraction expansion.

Multiplication by an exponential sequence

The exponential multiplication property is

$$z_0^n x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z/z_0), \qquad \text{ROC} = |z_0| R_x,$$

where the notation $|z_0|R_x$ indicates that the ROC is scaled by $|z_0|$ (that is, inner and outer radii of the ROC scale by $|z_0|$). All pole-zero locations are similarly scaled by a factor z_0: if $X(z)$ had a pole at $z = z_1$, then $X(z/z_0)$ will have a pole at $z = z_0 z_1$.

- If z_0 is positive and real, this operation can be interpreted as a shrinking or expanding of the z-plane — poles and zeros change along radial lines in the z-plane.

- If z_0 is complex with unit magnitude ($z_0 = e^{j\omega_0}$) then the scaling operation corresponds to a rotation in the z-plane by and angle ω_0. That is, the poles and zeros rotate along circles centered on the origin. This can be interpreted as a shift in the frequency domain, associated with modulation in the time domain by $e^{j\omega_0 n}$. If the Fourier transform exists, this becomes

$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

Exponential multiplication

The z-transform pair

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

can be used to determine the z-transform of $x[n] = r^n \cos(\omega_0 n) u[n]$. Since $\cos(\omega_0 n) = 1/2 e^{j\omega_0 n} + 1/2 e^{-j\omega_0 n}$, the signal can be rewritten as

$$x[n] = \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n].$$

Exponential multiplication

From the exponential multiplication property,

$$\frac{1}{2}(re^{j\omega_0})^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1/2}{1 - re^{j\omega_0}z^{-1}}, \qquad |z| > r$$

$$\frac{1}{2}(re^{-j\omega_0})^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1/2}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r,$$

so

$$\begin{aligned} X(z) &= \frac{1/2}{1 - re^{j\omega_0}z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned}$$

Differentiation

The differentiation property states that

$$nx[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

This can be seen as follows: since

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

we have

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

Evaluating convolution by z-trans.

The z-transforms of the signals $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$ are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

For $|a| < 1$, the z-transform of the convolution $y[n] = x_1[n] * x_2[n]$ is

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \qquad |z| > 1.$$

Using a partial fraction expansion,

$$Y(z) = \frac{1}{1 - a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \qquad |z| > 1,$$

so

$$y[n] = \frac{1}{1 - a}(u[n] - a^{n+1}u[n]).$$

Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

Relashionship with Laplace transform

Continuous-time systems and signals are usually described by the Laplace transform. Letting $z = e^{sT}$, where s is the complex Laplace variable

$$s = d + j\omega,$$

we have

$$z = e^{(d+j\omega)T} = e^{dT} e^{j\omega T}.$$

Therefore

$$|z| = e^{dT} \quad \text{and} \quad \sphericalangle z = \omega T = 2\pi f / f_s = 2\pi\omega/\omega_s,$$

where ω_s is the sampling frequency. As ω varies from ∞ to ∞, the s-plane is mapped to the z-plane:

- The $j\omega$ axis in the s-plane is mapped to the unit circle in the z-plane.
- The left-hand s-plane is mapped to the inside of the unit circle.
- The right-hand s-plane maps to the outside of the unit circle.

Examples and properties

- Complex conjugates zeros or poles:

$$1-2\rho\cos(\theta)z^{-1}+\rho^2 z^{-2}=0 \rightarrow z_{1,2}=\rho\left(\cos(\theta)\pm j\sin(\theta)\right)$$

- An infinite number of zeros approximates a (stable) pole:

$$\frac{1}{1-az^{-1}}=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n=\lim_{m\rightarrow+\infty}\sum_{n=0}^{m}\left(az^{-1}\right)^n=$$

$$=\lim_{m\rightarrow+\infty}\frac{1-a^{m+1}z^{-(m+1)}}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+\ldots$$

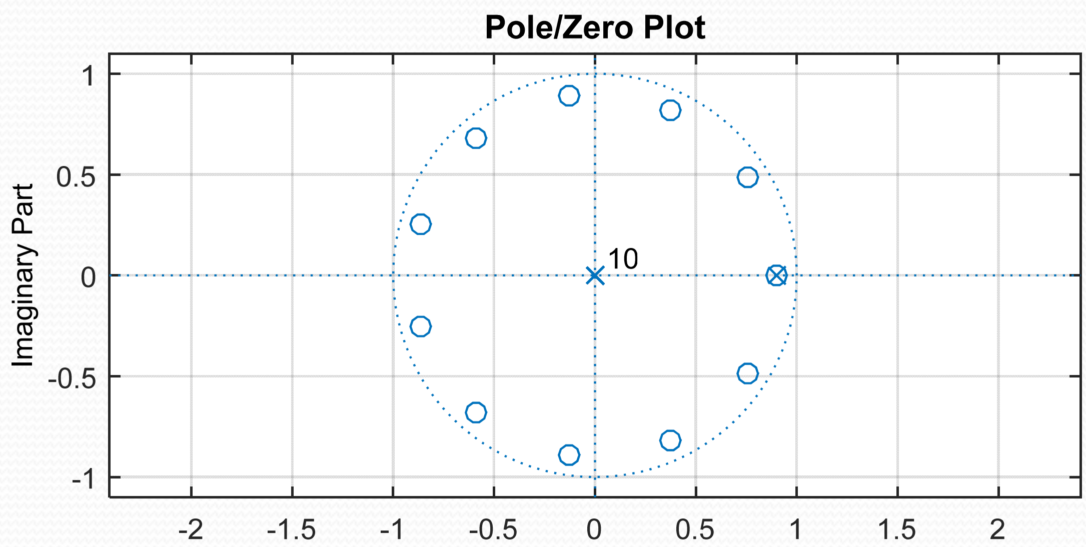
Examples and properties

$$\frac{1}{1-az^{-1}} = \lim_{m\to+\infty} \frac{1-a^{m+1}z^{-(m+1)}}{1-az^{-1}} \approx \left.\frac{1-a^{\bar{m}+1}z^{-(\bar{m}+1)}}{1-az^{-1}}\right|_{m=\bar{m}}$$

$$roots:\begin{cases} num: & z_{1,\bar{m}+1} = \sqrt[\bar{m}+1]{a^{\bar{m}+1}} \\ den: & z_1 = a \end{cases}$$

The pole is deleted by a zero in the same position

$$H(z) = \frac{1-0.9^{11}z^{-11}}{1-0.9z^{-1}}$$



All pass filters

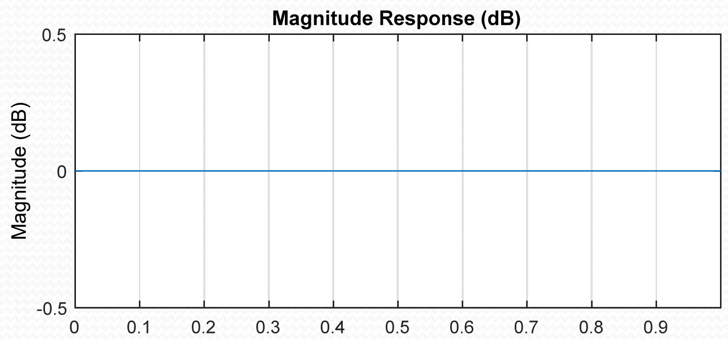
- An LTI with a zero and a pole in reciprocal conjugate position is the simplest All Pass Filter:

$$H(z) = \frac{c + z^{-1}}{1 + cz^{-1}}$$

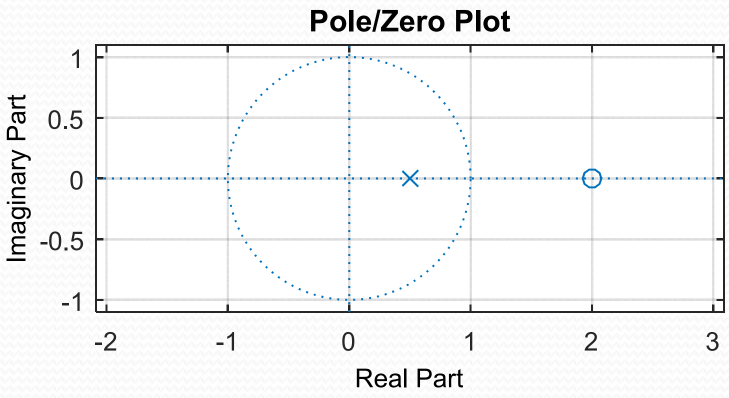
$$c = \frac{\tan(\pi f_c / f_s) - 1}{\tan(\pi f_c / f_s) + 1}$$

f_c = cut-off frequency

f_s = sampling frequency



- In this case $c = -\frac{1}{2}$



All pass filters

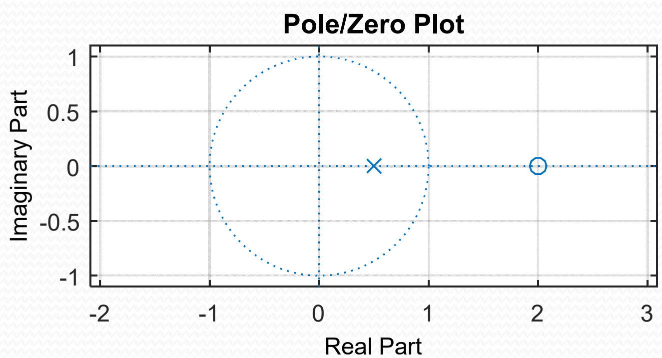
$$H(z)=\frac{-\frac{1}{2}+z^{-1}}{1-\frac{1}{2}z^{-1}}=-\frac{\frac{1}{2}-z^{-1}}{1-\frac{1}{2}z^{-1}}$$

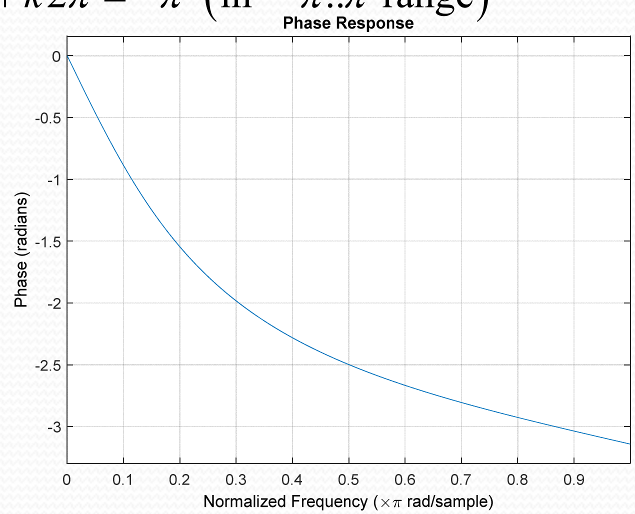
$$\measuredangle H(z)\big|_{z=e^{j\omega}}=\measuredangle(-1)+\measuredangle(z-2)\big|_{z=e^{j\omega}}-\measuredangle\left(z-\frac{1}{2}\right)\bigg|_{z=e^{j\omega}}=\pi+\operatorname{atan}2(\sin\omega,\cos\omega-2)-\operatorname{atan}2\left(\sin\omega,\cos\omega-\frac{1}{2}\right)+k2\pi$$

$$\measuredangle H(\omega=0)=\pi+\operatorname{atan}2(0,1-2)-\operatorname{atan}2\left(0,1-\frac{1}{2}\right)=\pi+\pi-0+k2\pi=0\ \ (\text{in }-\pi..\pi\text{ range})$$

$$\measuredangle H\left(\omega=\frac{\pi}{2}\right)=\pi+\operatorname{atan}2(1,-2)-\operatorname{atan}2\left(1,-\frac{1}{2}\right)\simeq\pi+2.68-2.03+k2\pi\simeq-2.49\ \ (\text{in }-\pi..\pi\text{ range})$$

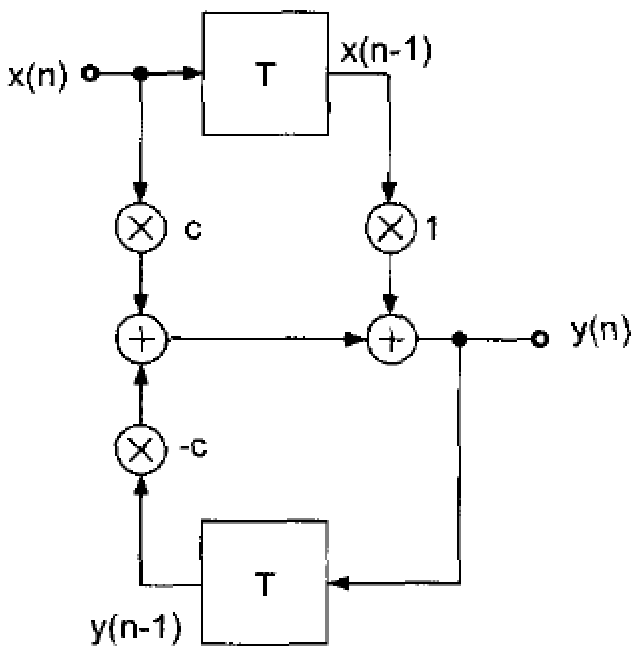
$$\measuredangle H(\omega=\pi)=\pi+\operatorname{atan}2(0,-1-2)-\operatorname{atan}2\left(0,-1-\frac{1}{2}\right)=\pi+\pi-\pi+k2\pi=-\pi\ \ (\text{in }-\pi..\pi\text{ range})$$

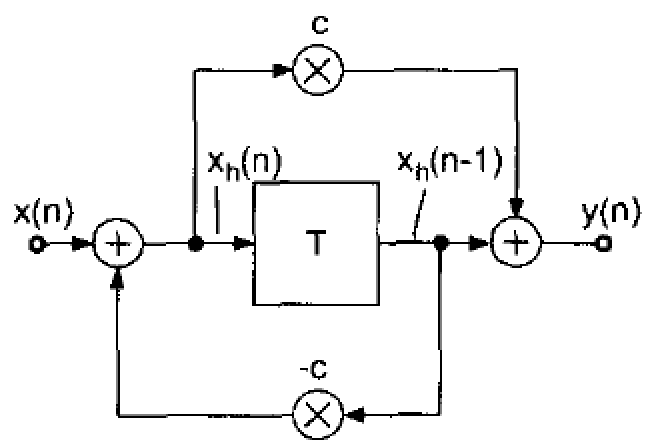




All pass filter implementation

$$y(n) \quad = \quad cx(n) + x(n-1) - cy(n-1),$$

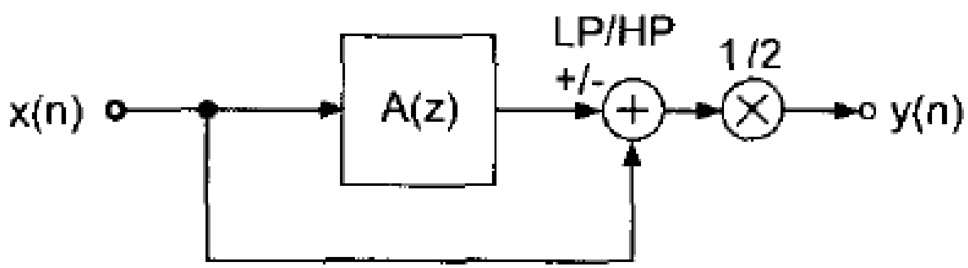




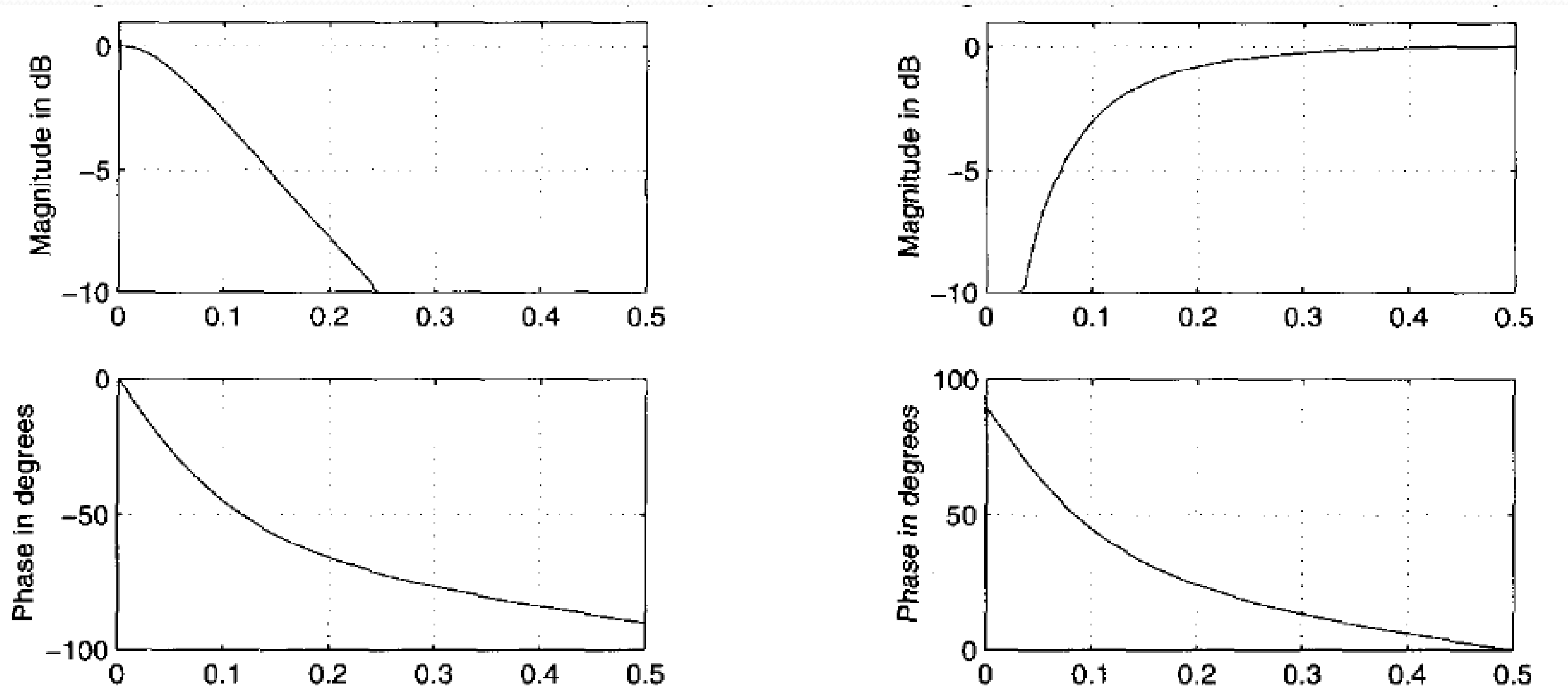
$$\begin{aligned} x_h(n) &= x(n) - cx_h(n-1) \\ y(n) &= cx_h(n) + x_h(n-1). \end{aligned}$$

First order low/high pass

$$
\begin{aligned}
H(z) &= \frac{1}{2}(1 \pm A(z)) \quad \text{(LP/HP +/-)} \\
A(z) &= \frac{z^{-1}+c}{1+cz^{-1}} \\
c &= \frac{\tan(\pi f_c/f_s)-1}{\tan(\pi f_c/f_s)+1},
\end{aligned}
$$



First order low/high pass



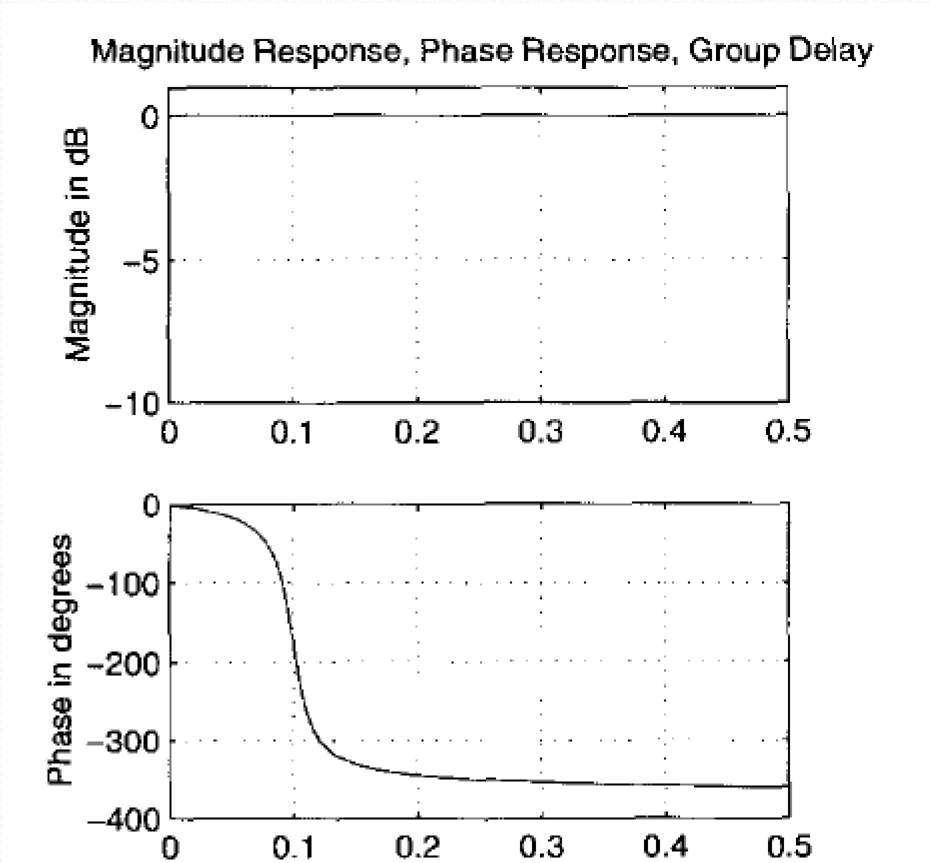
Second order all-pass filter

$$
\begin{aligned}
A(z) &= \frac{-c + d(1-c)z^{-1} + z^{-2}}{1 + d(1-c)z^{-1} - cz^{-2}} \\
c &= \frac{\tan(\pi f_b/f_s) - 1}{\tan(\pi f_b/f_s) + 1} \\
d &= -\cos(2\pi f_c/f_s).
\end{aligned}
$$

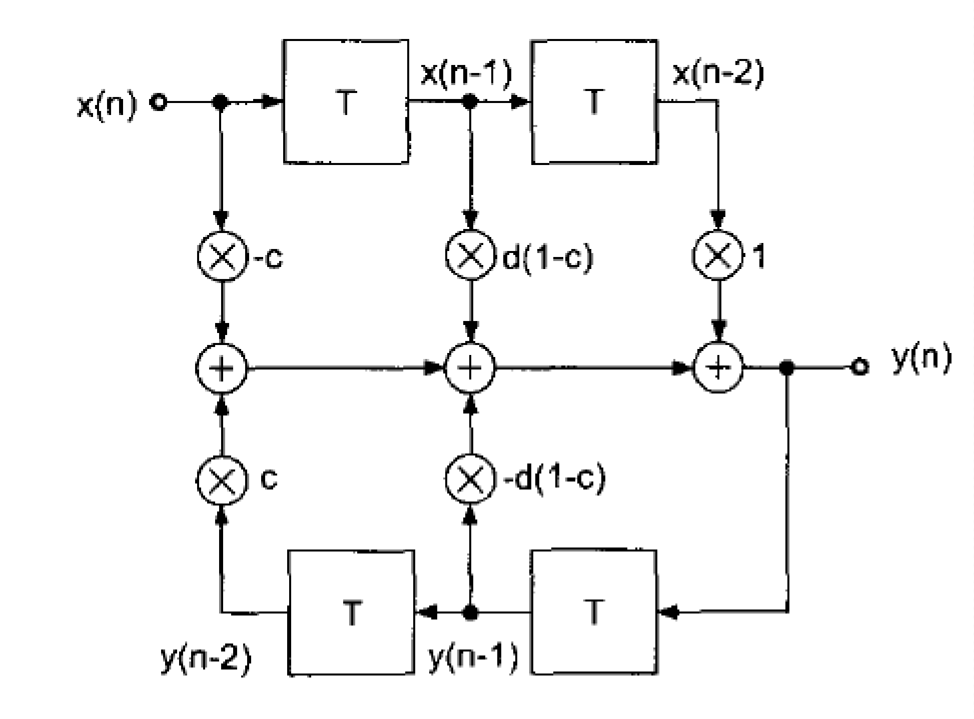
- *d* adjusts the cut-off frequency while *c* the bandwidth.

$$
\begin{aligned}
y(n) &= -cx(n) + d(1-c)x(n-1) + x(n-2) \\
&\quad -d(1-c)y(n-1) + cy(n-2)
\end{aligned}
$$

Second order all-pass filter

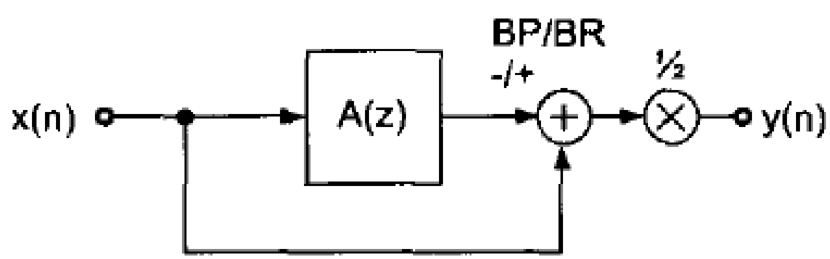


Implementation of a second order all pass filter

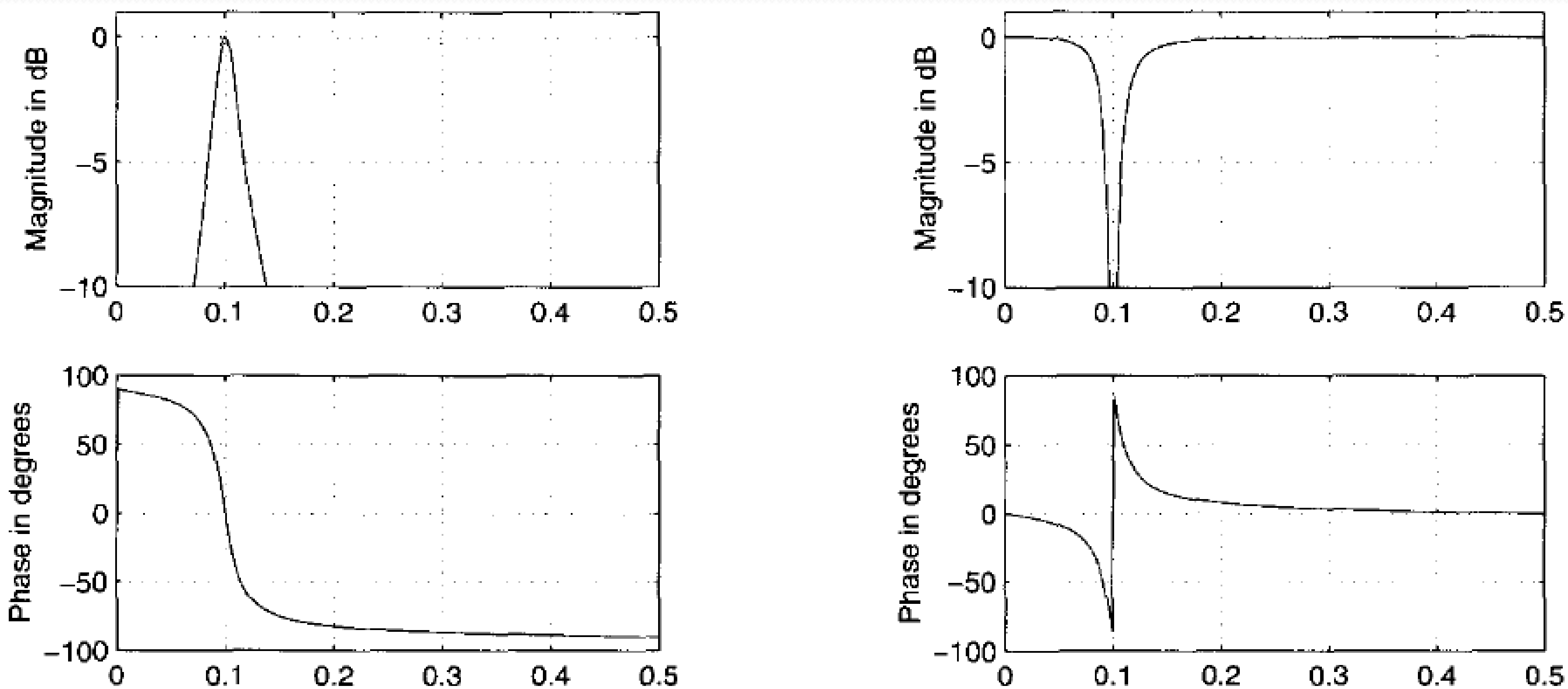


Second order band pass/band reject

$$
\begin{aligned}
H(z) &= \frac{1}{2}\left[1 \mp A(z)\right] \quad (\mathrm{BP/BR}\ \text{-}/+) \\
A(z) &= \frac{-c + d(1-c)z^{-1} + z^{-2}}{1 + d(1-c)z^{-1} - cz^{-2}} \\
c &= \frac{\tan(\pi f_b/f_s) - 1}{\tan(2\pi f_b/f_s) + 1} \\
d &= -\cos(2\pi f_c/f_s),
\end{aligned}
$$



Second order band pass/band reject



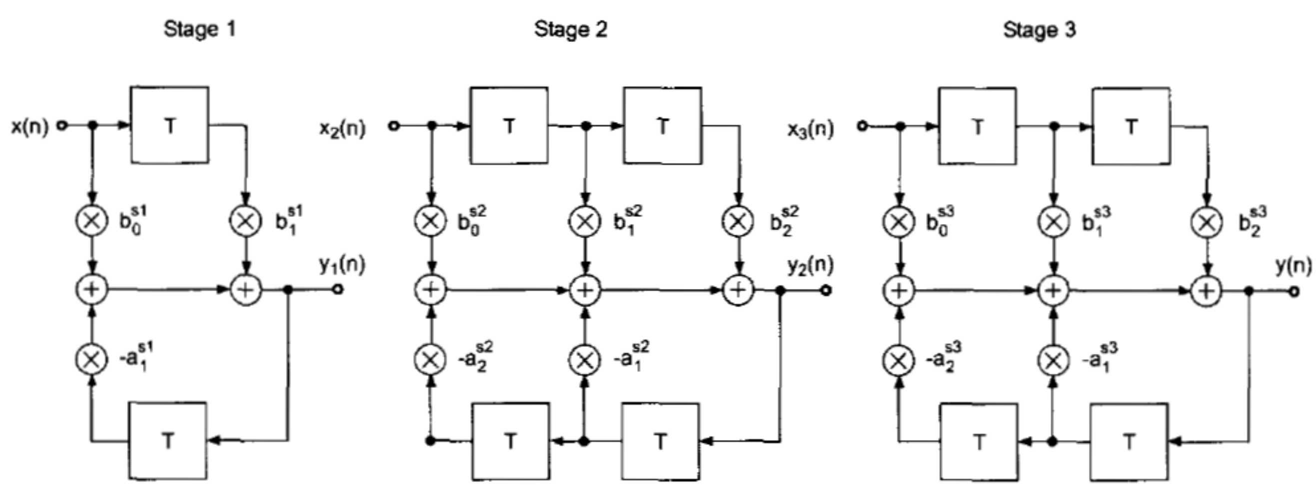
Series connection of first and second order filters

- If several filters are necessary for spectrum shaping a series connection of first and second order filters is performed

$$H_{\text{1st-order}}(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

$$H_{\text{2nd-order}}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \cdot H_2(z) \cdot H_3(z).$$



Linear phase filters

- A linear-phase filter is typically used when a causal filter is needed to modify a signal's magnitude-spectrum while preserving the signal's time-domain waveform as much as possible. Linear-phase filters have a symmetric impulse response, e.g.,

$$h(n) = h(N-1-n) \quad n = 0,1,\ldots,N-1$$

- every real symmetric impulse response corresponds to a real frequency response times a linear phase: $e^{-j\alpha\omega}$ where

$$\alpha = \frac{N-1}{2}$$

Linear phase filters

- α is the *slope* of the linear phase.
- The filter phase has the form:

$$\Theta(\omega) = -\alpha\omega$$

- Phase delay will be:

$$P(\omega) = -\frac{\Theta(\omega)}{\omega} = -\frac{-\alpha\omega}{\omega} = -\alpha$$

- Group delay will be:

$$G(\omega) = -\frac{\partial}{\partial\omega}\Theta(\omega) = \alpha$$