

# The Z-Transform

## Lesson 3

# The z-transform

The z-transform of a sequence  $x[n]$  is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The z-transform can also be thought of as an operator  $\mathcal{Z}\{\cdot\}$  that transforms a sequence to a function:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z).$$

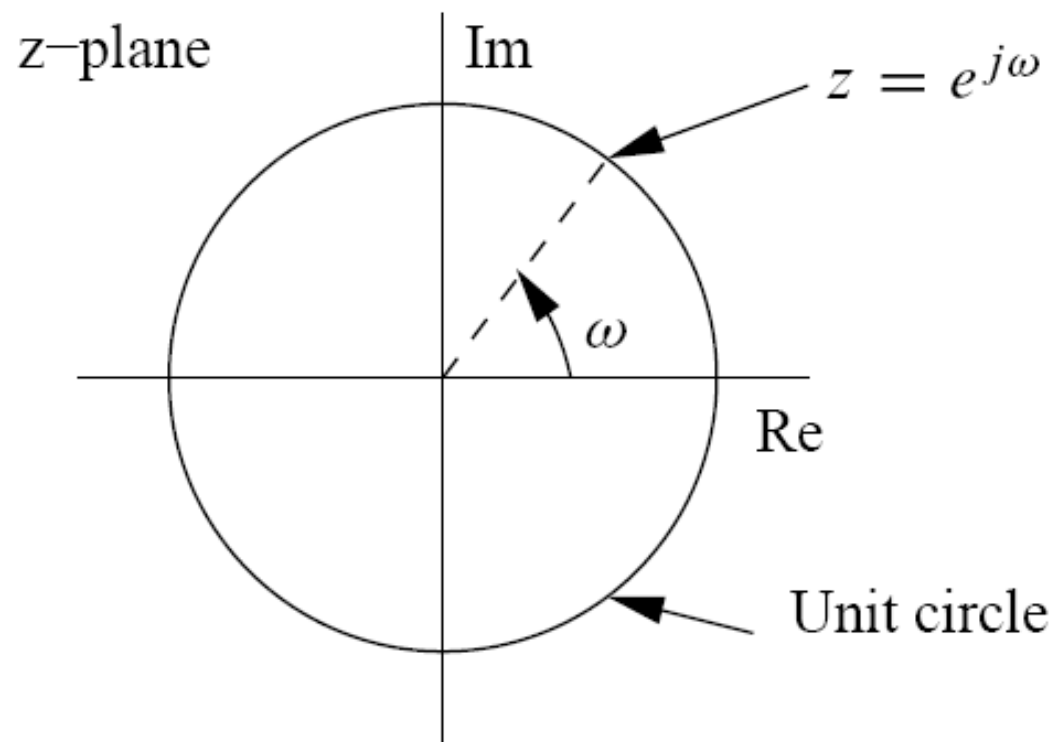
In both cases  $z$  is a continuous complex variable.

We may obtain the DTFT from the z-transform by making the substitution  $z = e^{j\omega}$ . This corresponds to restricting  $|z| = 1$ . Also, with  $z = re^{j\omega}$ ,

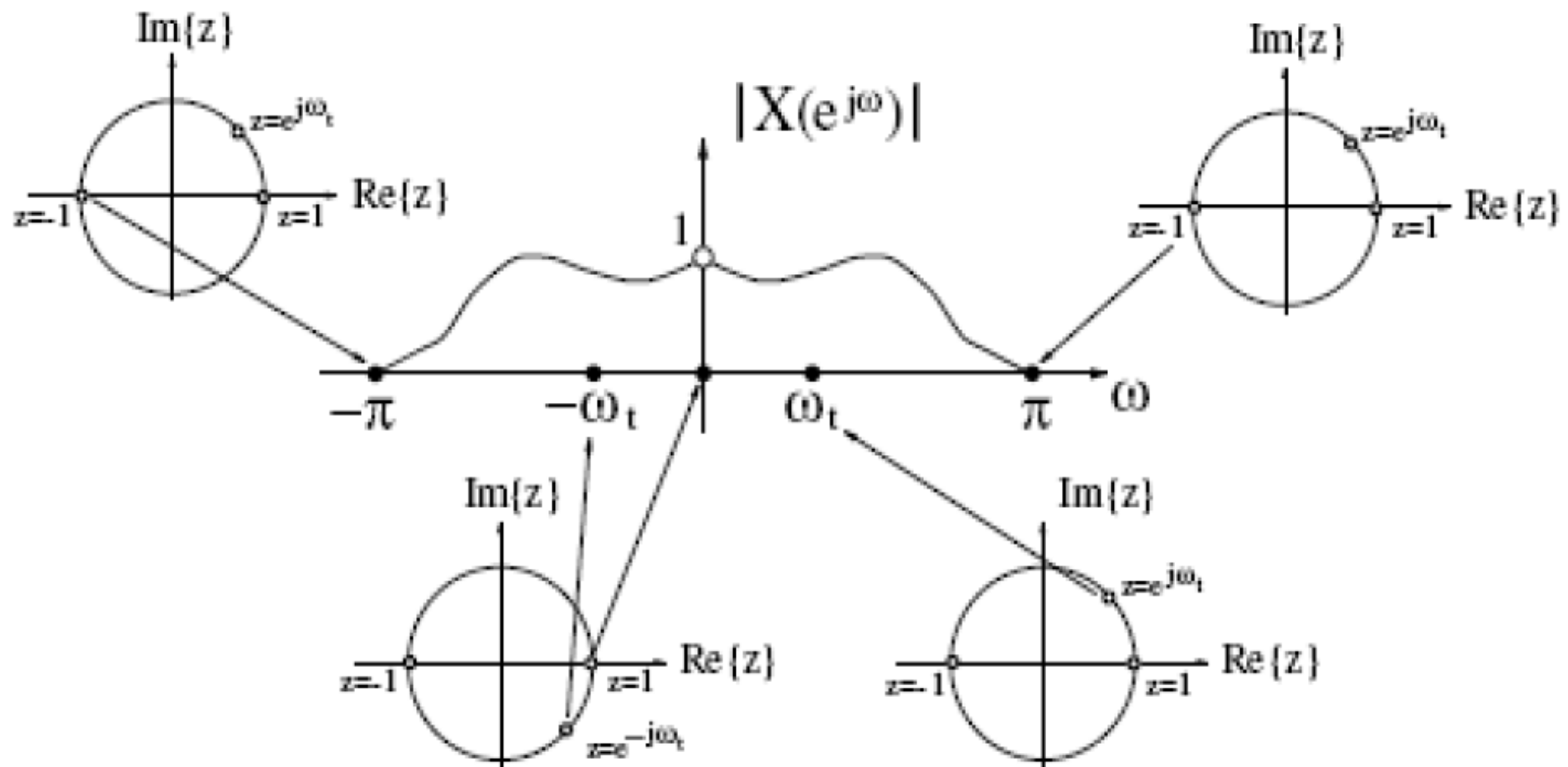
# Fourier and z transform

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n}) e^{-j\omega n}.$$

That is, the z-transform is the DTFT of the sequence  $x[n]r^{-n}$ . For  $r = 1$  this becomes the DTFT of  $x[n]$ . The Fourier transform therefore corresponds to the z-transform evaluated on the unit circle:



# Relation between Z-Transform and DTFT





# Region of convergence

The inherent periodicity in frequency of the Fourier transform is captured naturally under this interpretation.

The Fourier transform does not converge for all sequences — the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of  $z$ . The set of values of  $z$  for which the z-transform converges is called the **region of convergence (ROC)**.

The Fourier transform of  $x[n]$  exists if the sum  $\sum_{n=-\infty}^{\infty} |x[n]|$  converges. However, the z-transform of  $x[n]$  is just the Fourier transform of the sequence  $x[n]r^{-n}$ . The z-transform therefore exists (or converges) if

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

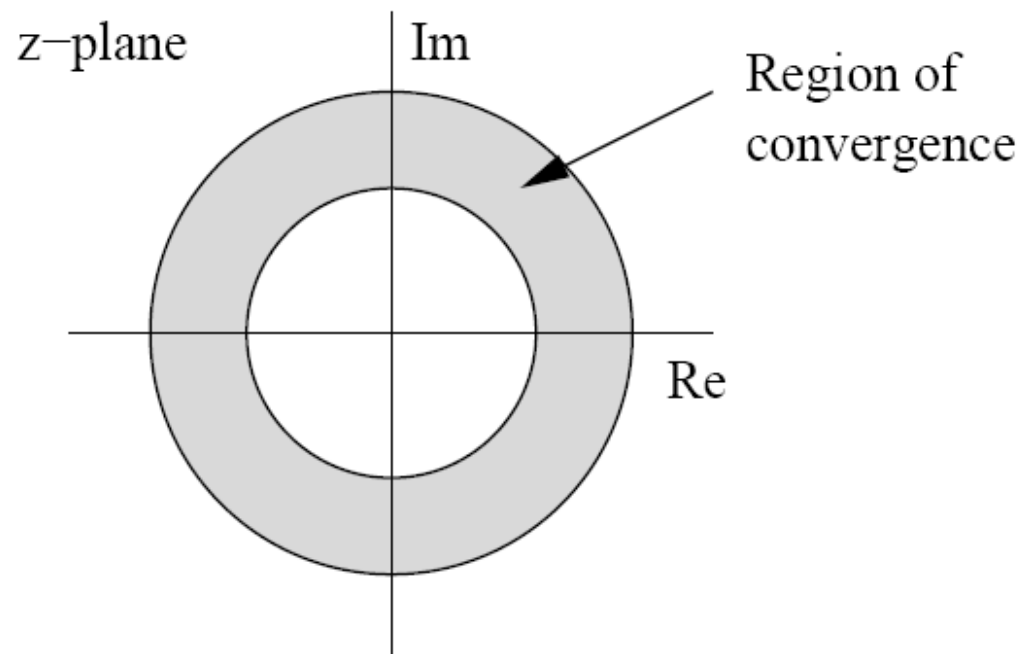
# Region of Convergence

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:

# Ring of Convergence in the z-plane



In specific cases the inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle  $|z| = 1$ , then the Fourier transform will converge.



# Examples of z-transform

- The z-transform of the bilateral sequence  $x(n) = 2\delta(n+1) + \delta(n) + 4\delta(n-2)$  is:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} [2\delta(n+1) + \delta(n) + 4\delta(n-2)] z^{-n} = \\ &= 2 \sum_{n=-\infty}^{\infty} \delta(n+1) z^{-n} + \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} + 4 \sum_{n=-\infty}^{\infty} \delta(n-2) z^{-n} = \\ &= 2z + 1 + 4z^{-2} \end{aligned}$$



# Geometric series

- The general formula for converging geometric series is:

$$\sum_{n=0}^k q^n = \frac{1 - q^{k+1}}{1 - q}$$

- For infinite series the convergence request is:  $|q| < 1$

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1 - q}.$$

# Examples of the Z- transform

- The z transform for the step function  $u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}$$

$$x(n) = \delta(n) \quad X(z) = \sum_{n=-\infty}^{+\infty} \delta(n)z^{-n} = 1$$

Causal

Anticausal

$$x(n) = \delta(n-k) \quad k > 0$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta(n-k)z^{-n} = z^{-k}$$

$$x(n) = \delta(n+k) \quad k > 0$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta(n+k)z^{-n} = z^k$$

# Examples of the Z- transform

$$x_1(n) = \{1, 2, 5, 7, 0, 1\}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n} = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 1z^{-5}$$

$$x(n) = \{1, 2, 5, 7, 0, 1\}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x_1(n) z^{-n} = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$$



# Finite causal sequence z-transform

- Given the following sequence:
- $x(n) = \alpha^n [u(n) - u(n - N)]$ ,
- Where  $N$  is an integer and  $\alpha$  is a real constant.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{N-1} \alpha^n z^{-n} = \\ &= \sum_{n=0}^{N-1} (\alpha z^{-1})^n = \frac{1 - \alpha^N z^{-N}}{1 - \alpha z^{-1}} \end{aligned}$$

# Example of a finite causal sequence

- **Poles** are the roots of the *denominator*
- **Zeros** are the roots of the *numerator*

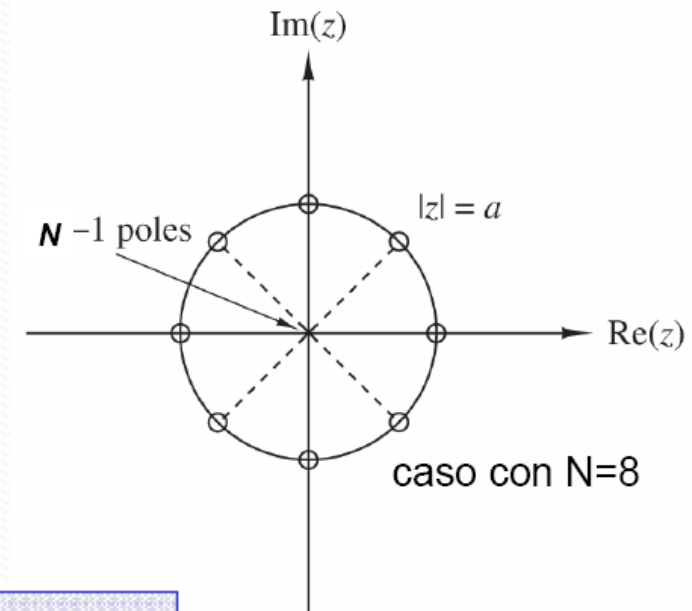
$$X(z) = \frac{1 - \alpha^N z^{-N}}{1 - \alpha z^{-1}} = z^{-N+1} \frac{z^N - \alpha^N}{z - \alpha}$$

- The  $X(z)$  has a pole of order  $N - 1$  in the origin and  $N - 1$  zeros (the  $z = \alpha$  root pole at the denominator is compensated by a zero in the same position).



# Example of a finite causal sequence

- The polynomial  $N: z^N - \alpha^N$  has  $N$  zeros uniformly distributed along the circle of radius  $\alpha$ .
- The roots of the polynomial are at these complex pulsations.



$$z = \alpha e^{j\frac{2\pi}{N}k} \quad k = 0, \dots, N-1$$

$$X(z) = \frac{1 - \alpha^N z^{-N}}{1 - \alpha z^{-1}} = z^{-N+1} \frac{z^N - \alpha^N}{z - \alpha}$$

$$z = \alpha e^{j\frac{2\pi}{N}k} \quad k = 1, \dots, N-1$$



# Example of an infinite causal sequence

- Find the z-transform of the following sequence:

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$X(z) = \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

- It is a geometric series of ratio:  $\frac{1}{2} z^{-1}$  converging at:

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

# Rational Z transform

- For most common cases  $X(z)$  is the ratio between two polynomials:

$$X(z) = N(z)/D(z)$$

- Where  $N(z)$  e  $D(z)$  are two polynomials of variable  $z^{-1}$ , of degree  $p_n$  and  $p_d$  respectively.
- The extended notation of the two polynomials are  $N(z)$  and  $D(z)$ :

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{p_n} z^{-p_n}}{a_0 + a_1 z^{-1} + \dots + a_{p_d} z^{-p_d}}$$



# Rational Z transform

- The z-transform can be expressed through the factorization of both numerator and denominator:

$$X(z) = \frac{b_0}{a_0} \left( z^{p_d - p_n} \right) \frac{\prod_{i=1}^{p_n} (z - c_i)}{\prod_{i=1}^{p_d} (z - d_i)}$$



# LTI systems analysis by the z-transform

- Time discrete LTI systems can be described as finite differences linear equations with constant coefficients.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_M y(n-M) + b_0 x(n) + b_1 x(n-1) + \dots + b_N x(n-N)$$

- Applying the DTFT to each term:

$$Y(e^{j\omega}) = -a_1 Y(e^{j\omega}) e^{-j\omega} - \dots - a_M Y(e^{j\omega}) e^{-j\omega M} + b_0 X(e^{j\omega}) + b_1 X(e^{j\omega}) e^{-j\omega} + \dots + b_N X(e^{j\omega}) e^{-j\omega N}$$

- Remember:

$$F(f(x - x_0)) = e^{-j2\pi u x_0} F(u)$$

# LTI system analysis by the z-transform

- Gathering all the terms in  $Y(e^{j\omega})$  and  $X(e^{j\omega})$  we obtain

$$Y(e^{j\omega}) \left( 1 + a_1 e^{-j\omega} + \dots + a_M e^{-j\omega M} \right) = X(e^{j\omega}) \left( b_0 + b_1 e^{-j\omega} + \dots + b_N e^{-j\omega N} \right)$$

- Thanks to the convolution theorem:

$$y(n) = x(n) * h(n) \Leftrightarrow Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

- The frequency response  $H(e^{j\omega})$  for an LTI system defined by finite differences equations can be defined as:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b_0 + b_1 e^{-j\omega} + \dots + b_N e^{-j\omega N}}{1 + a_1 e^{-j\omega} + \dots + a_M e^{-j\omega M}}$$



# LTI system analysis by the z-transform

- Thanks to the relation between the DTFT and the z-transform :

$$\sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \underbrace{\left( x(n) \rho^{-n} \right)}_{\text{DTFT } (x(n) \rho^n)} e^{-j\omega n}$$

- Through the application of the convolution theorem the response of the complex frequency system Z is:

$$Y(z) = Z[x(n) * h(n)] = X(z)H(z)$$



# LTI system analysis by the z-transform

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(b_0 + b_1 e^{-j\omega} + \dots + b_N e^{-j\omega N})}{(1 + a_1 e^{-j\omega} + \dots + a_M e^{-j\omega M})}$$

- We know that the transfer function  $H(e^{j\omega})$  is a rational function, so that the transfer function  $H(z)$  is also a rational function like:  $N(z)/D(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 + b_1 z^{-1} + \dots + b_N z^{-N})}{(1 + a_1 z^{-1} + \dots + a_M z^{-M})}$$

- That corresponds to the finite difference equation.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_M y(n-M) + b_0 x(n) + b_1 x(n-1) + \dots + b_N x(n-N)$$

# LTI systems

- In the set of LTI system, described through the difference equations we can define two kinds of systems.
- FIR, Finite Impulse Response filters: non recursive systems where the output  $y(n)$  has a dependence just from the input signal  $x(n)$ :

$$y(n) = \sum_{k=0}^N b_k x(n-k)$$



# LTI systems

- IIR, Infinite Impulse Response Filters: systems where the outputs  $y(n)$  depend both from inputs and from outputs themselves :

$$y(n) = \sum_{k=0}^N b_k x(n-k) - \sum_{j=1}^M a_j y(n-j)$$

- A sub-set of this kind of systems concern only recursive filters, i.e. where the output depends just from the actual input and from previous outputs.

$$y(n) = x(n) - \sum_{j=1}^M a_j y(n-j)$$



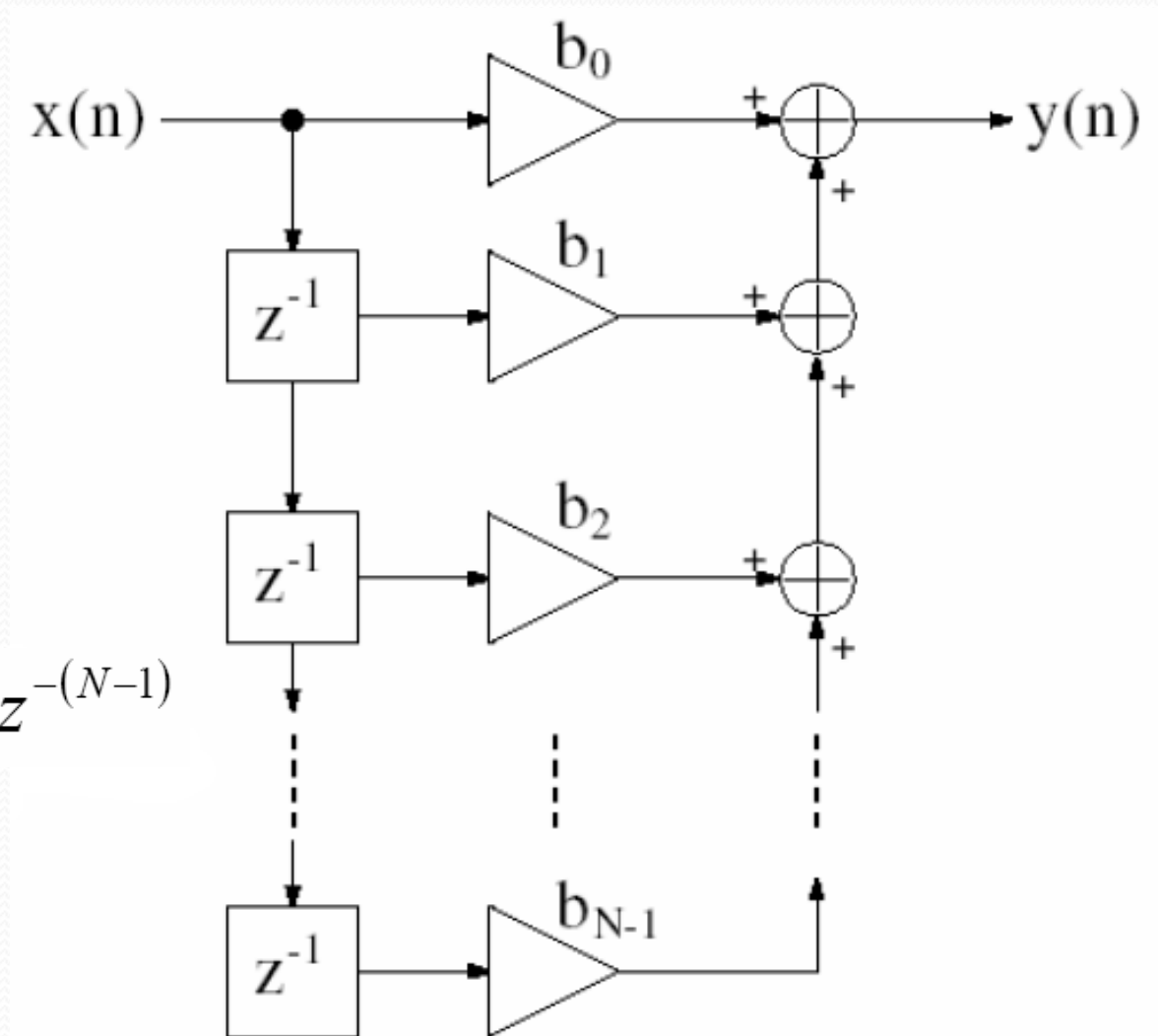
# FIR and their z-transform

- A causal LTI non recursive FIR filter

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

- It can be described by the equation:

$$H(z) = b_0 + b_1 z^{-1} + \dots + b_{N-1} z^{-(N-1)}$$



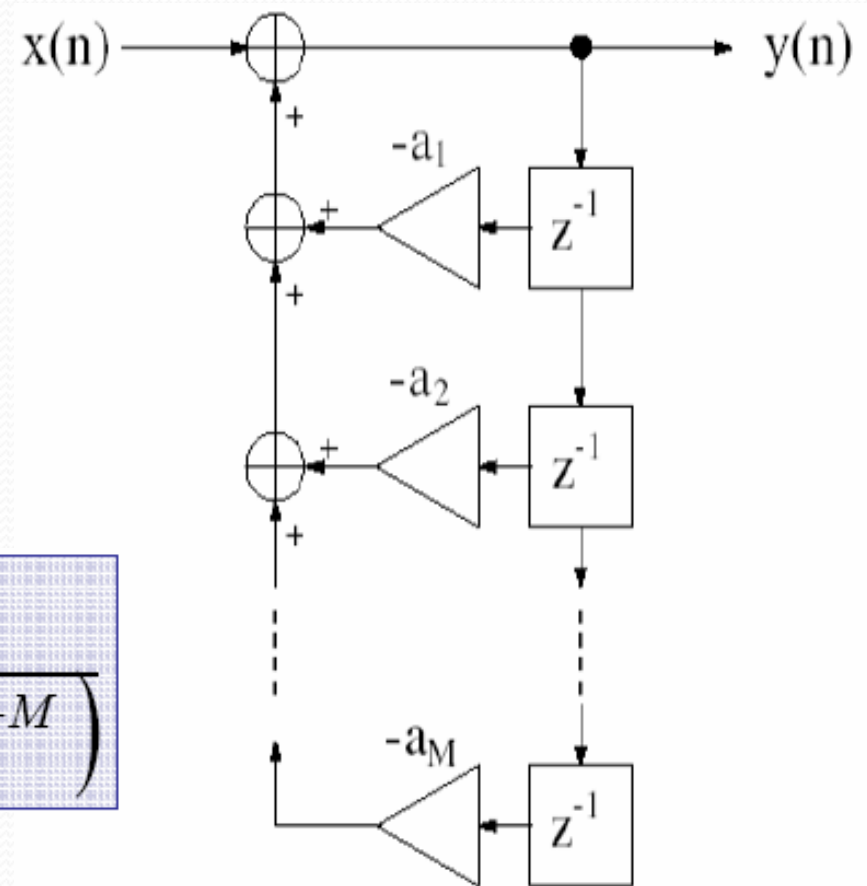
# Pure recursive IIR filter

- An LTI causal system made of a pure IIR is a filter like:

$$y(n) = x(n) - \sum_{j=1}^M a_j y(n-j)$$

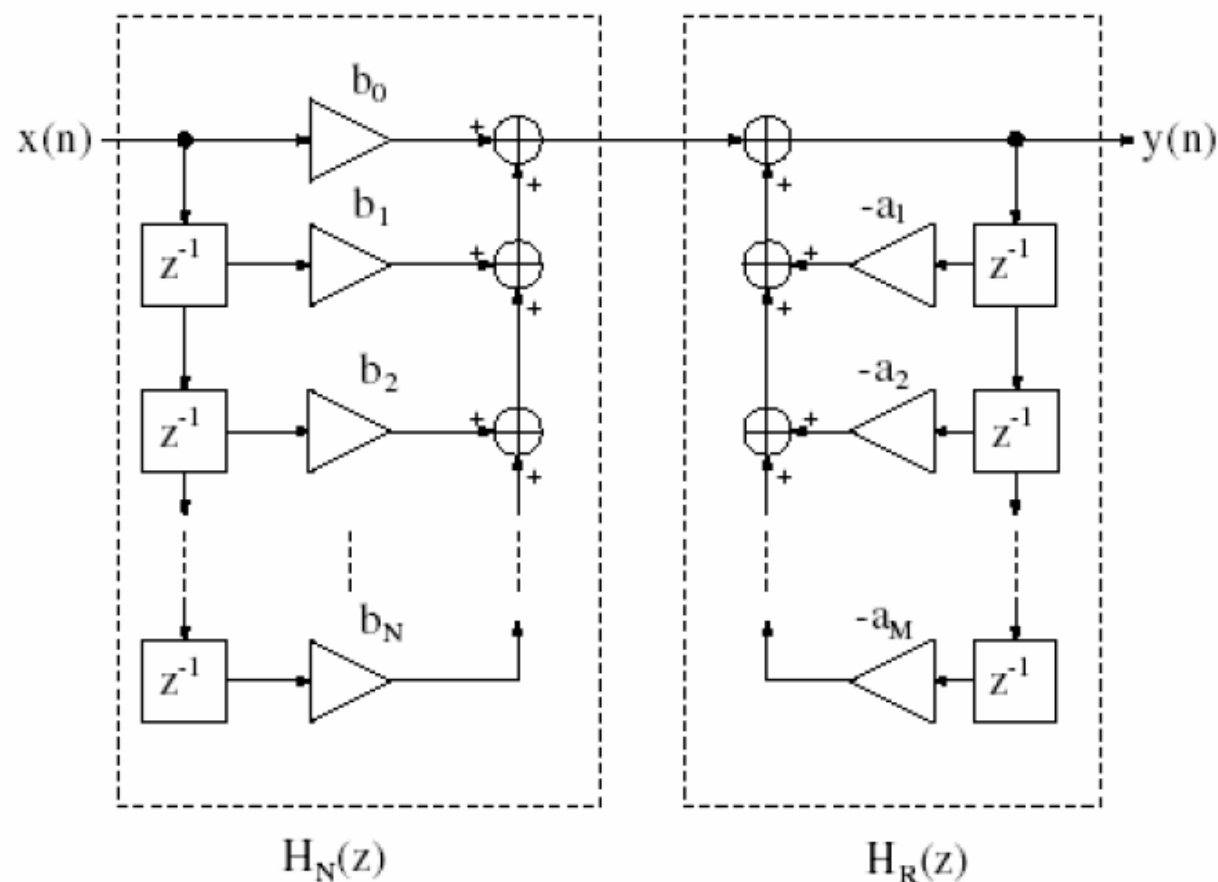
- It can be described using the following equation.

$$H(z) = \frac{1}{(1 + a_1 z^{-1} + \dots + a_M z^{-M})}$$



# LTI general case for a LTI system

$$H(z) = \frac{(b_0 + b_1 z^{-1} + \dots + b_N z^{-N})}{(1 + a_1 z^{-1} + \dots + a_M z^{-M})}$$





# LTI and z-transform

- The transfer function  $H(z)$  can be thought in terms of the roots of the polynomials of the numerator and

$$H(z) = \frac{N(z)}{D(z)} = Kz^{M-N} \frac{(z - c_1)(z - c_2) \dots (z - c_N)}{(z - d_1)(z - d_2) \dots (z - d_M)}$$

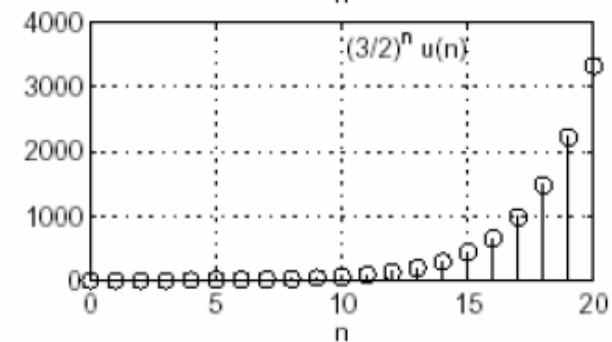
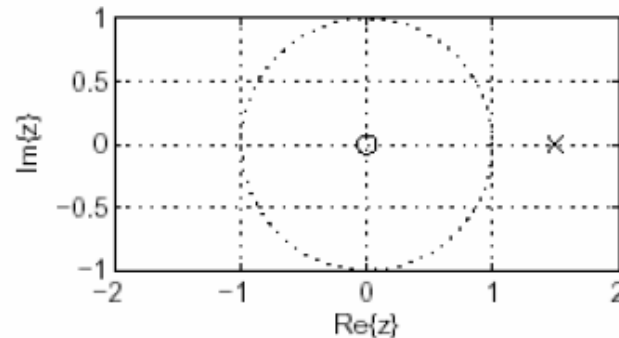
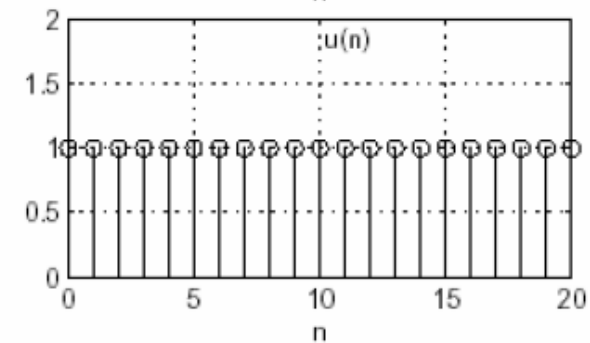
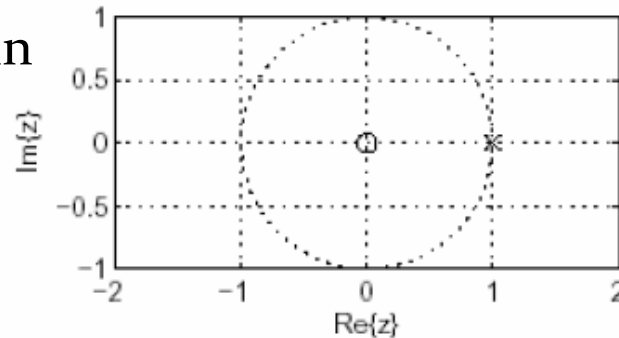
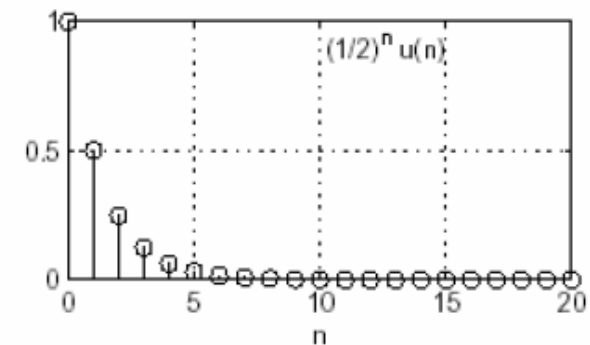
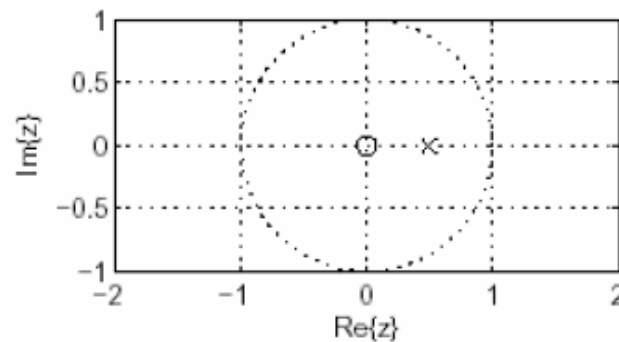
- The output of a LTI system is described and analyzed by its zeros and poles.
- For a causal system the number of zeros cannot be larger than the number of poles .

$$x(n) = \alpha^n u(n)$$

$$X(z) = \frac{1}{1 - \alpha z^{-1}} = z \frac{1}{z - \alpha}$$

$$\alpha > 0$$

1 pole in  $\alpha$   
1 zero in the origin

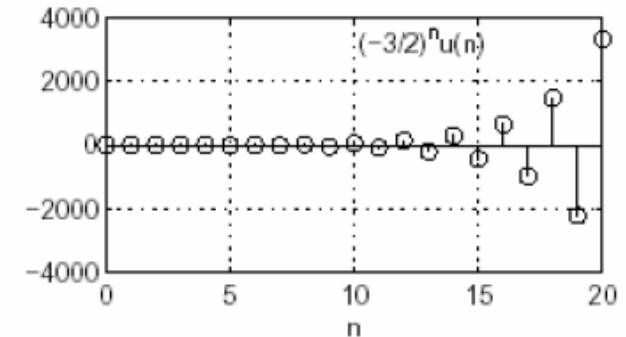
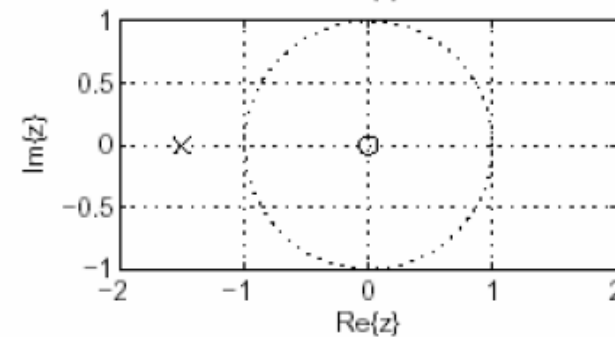
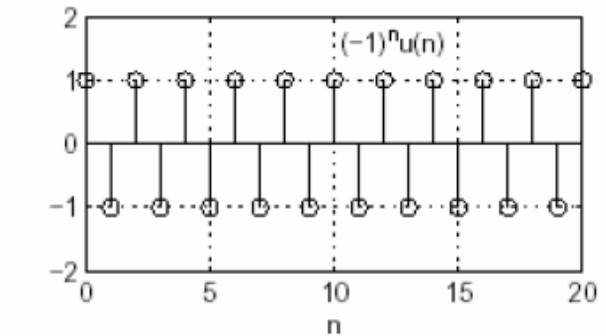
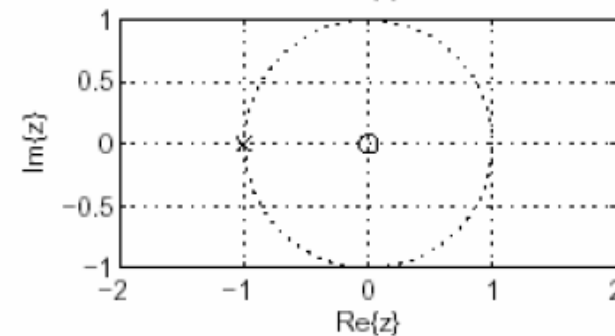
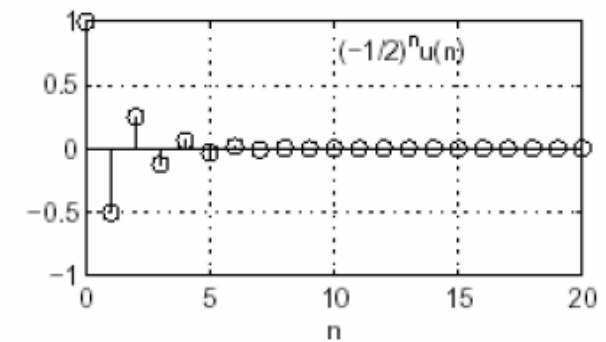
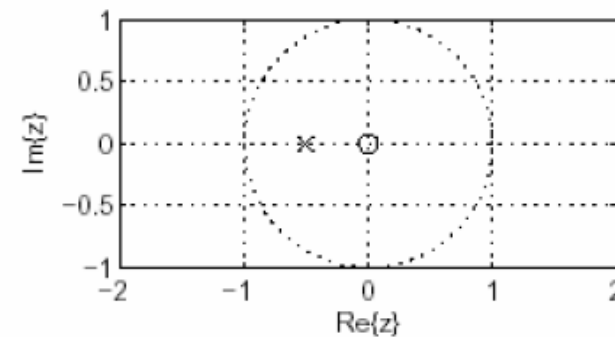


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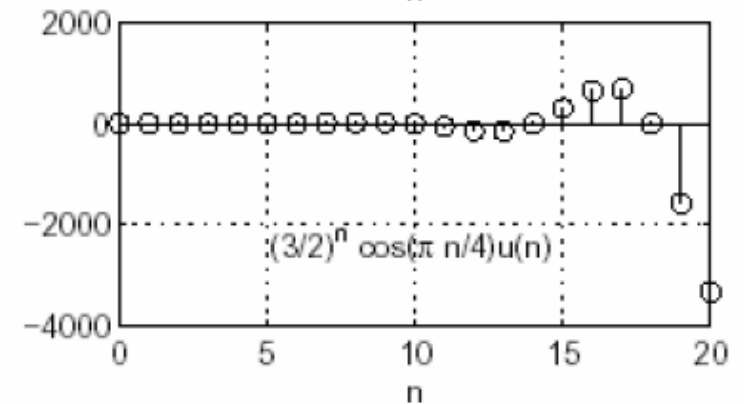
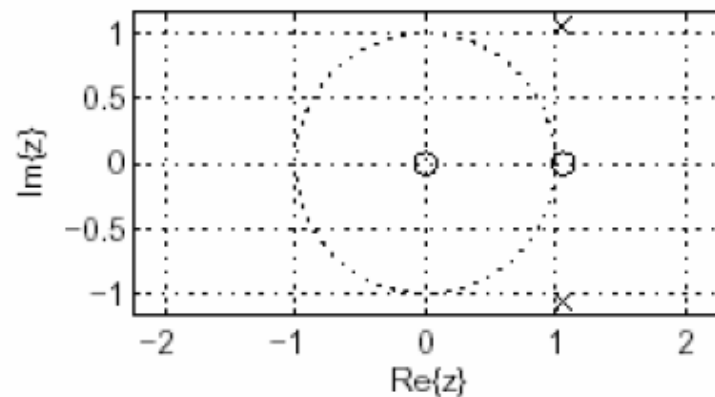
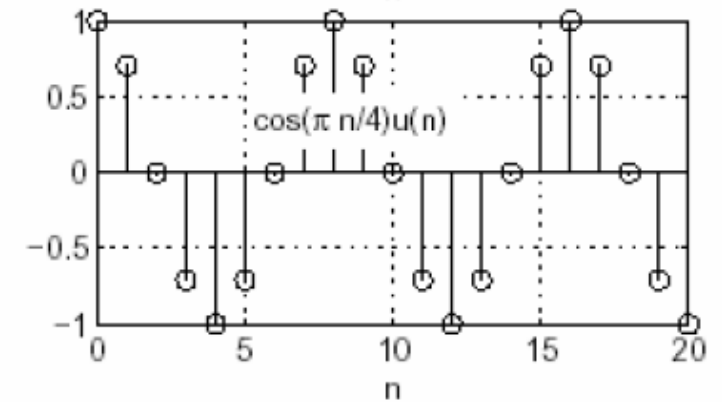
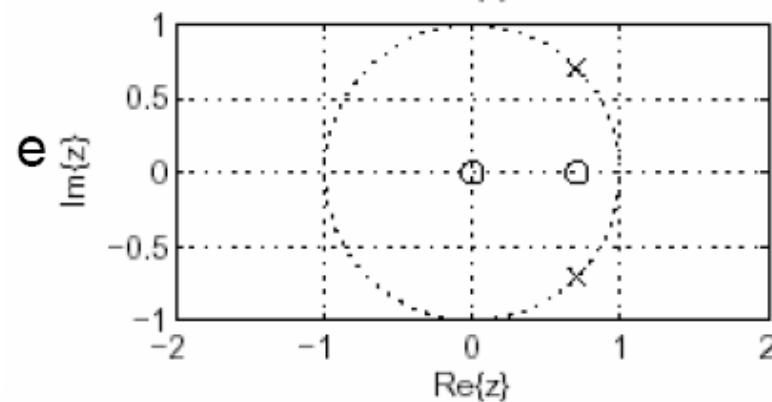
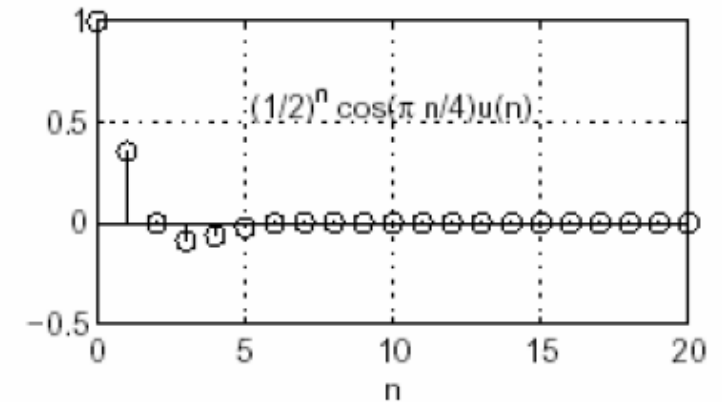
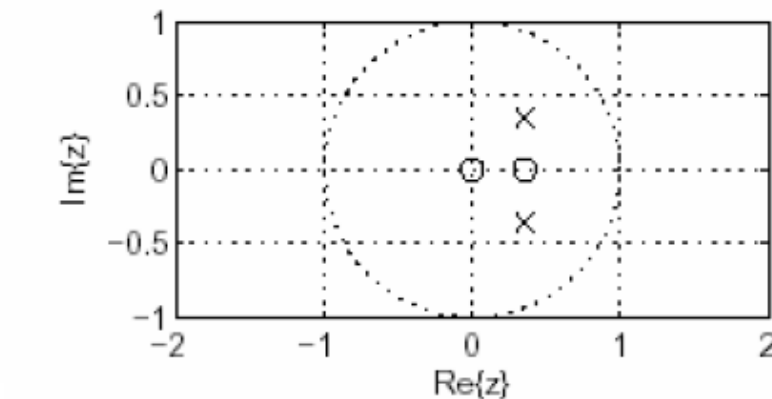


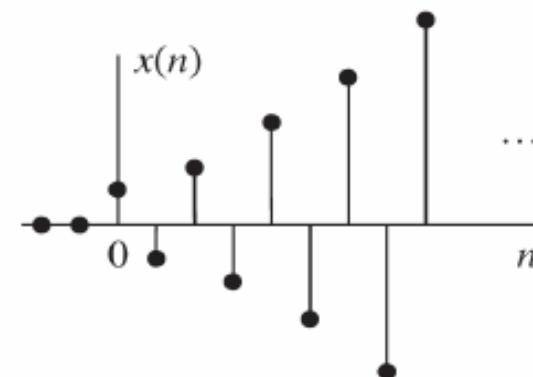
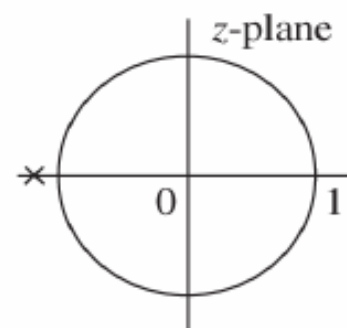
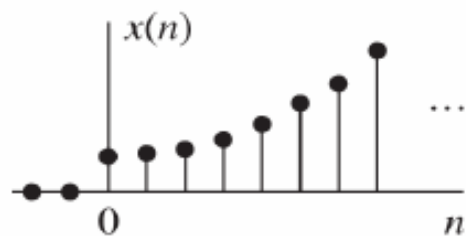
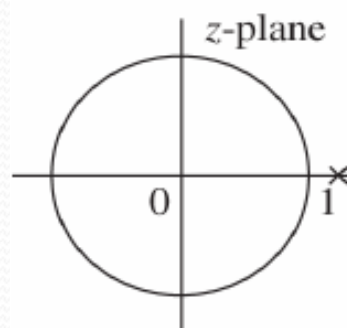
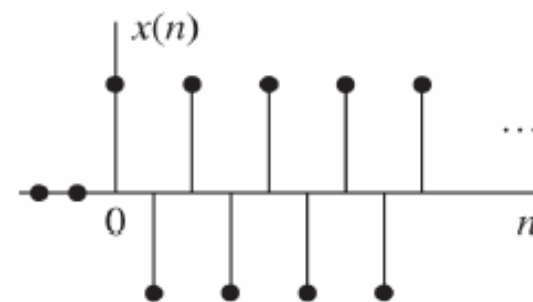
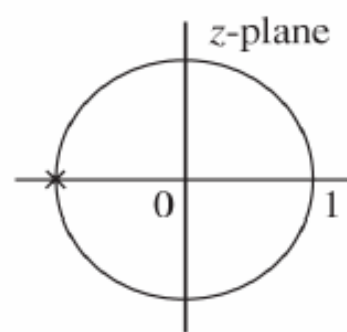
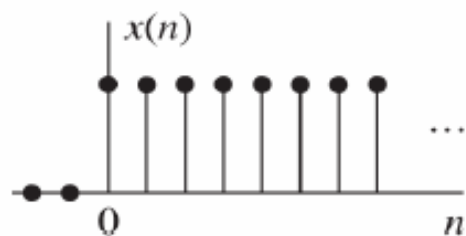
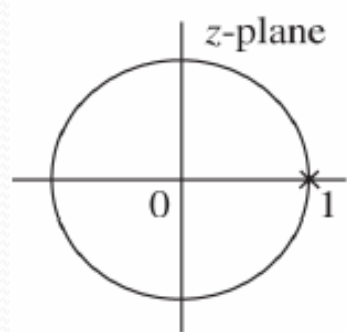
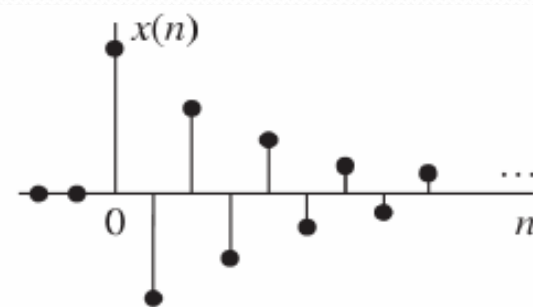
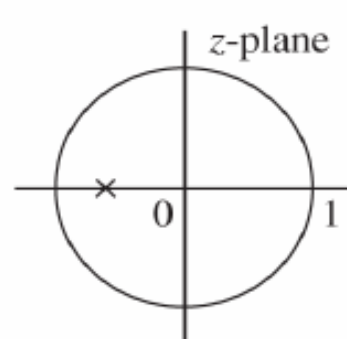
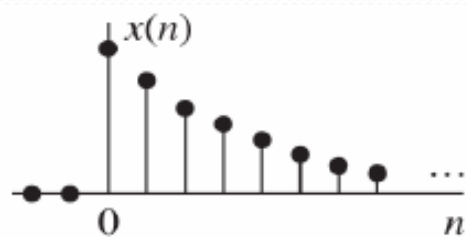
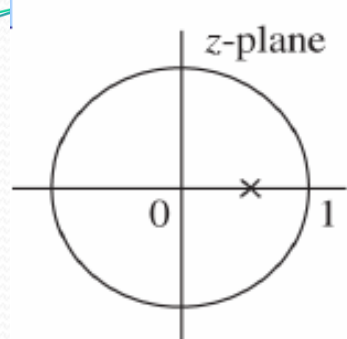


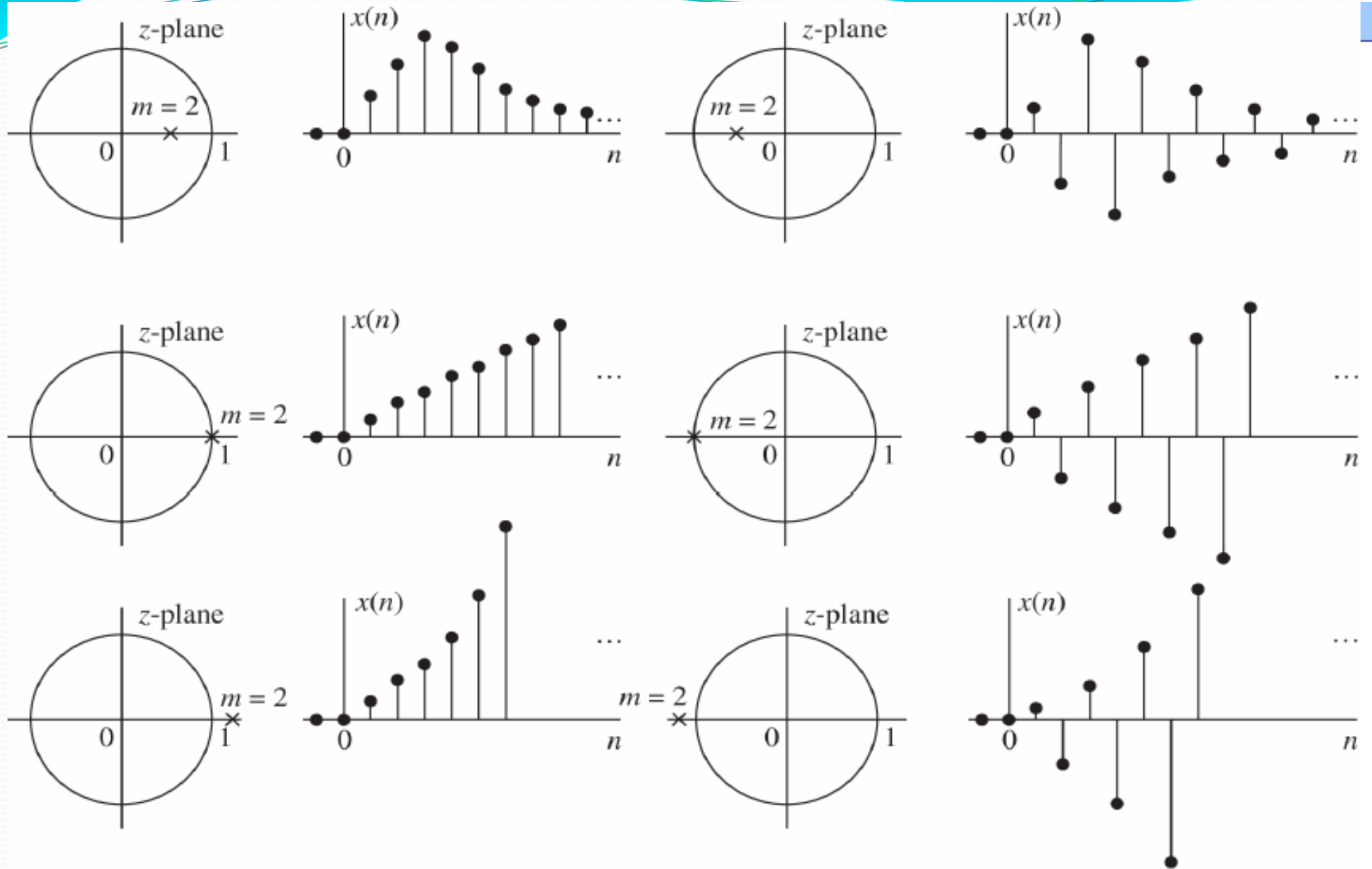
# Complex and conjugate poles

$$\alpha > 0$$

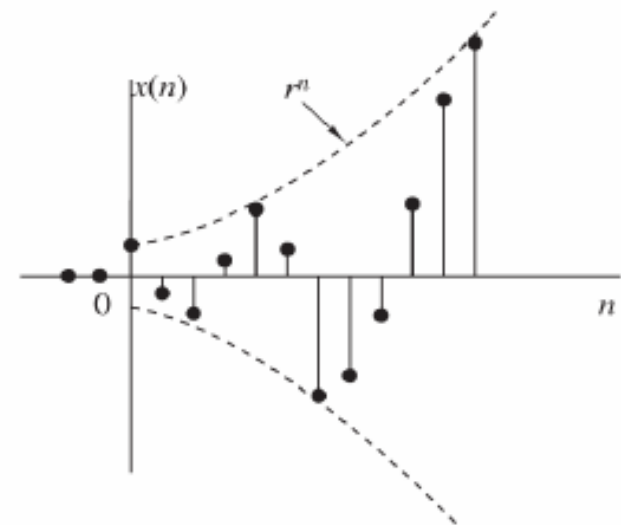
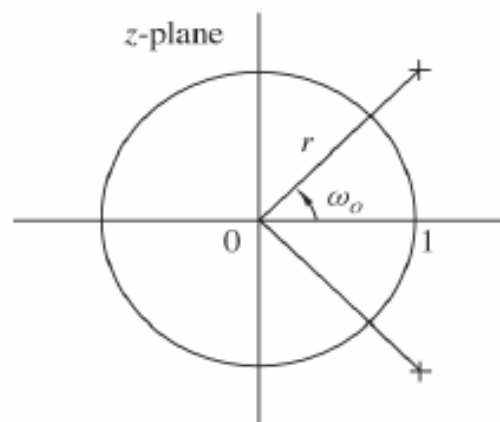
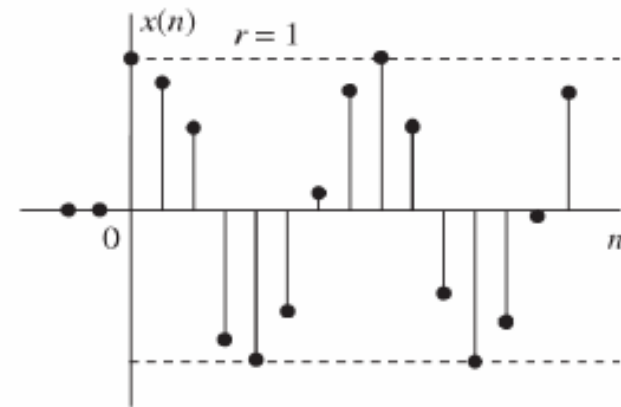
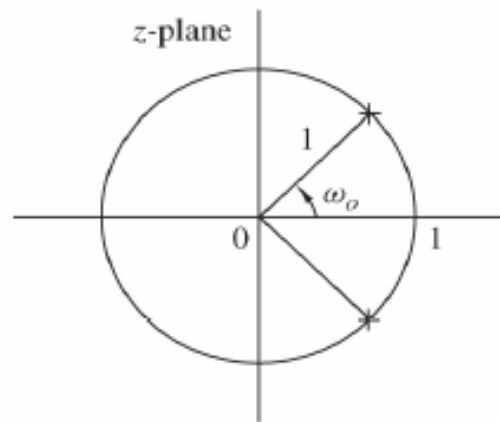
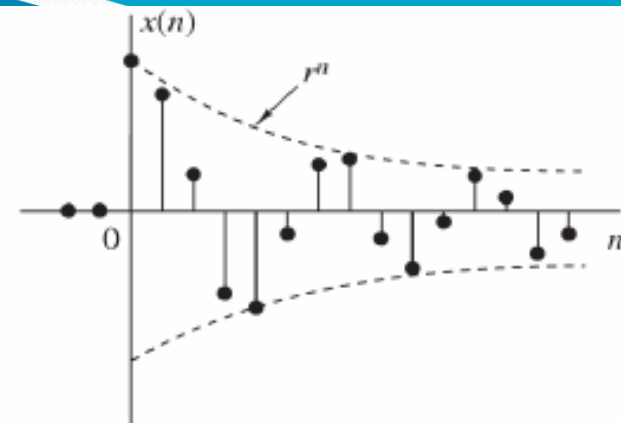
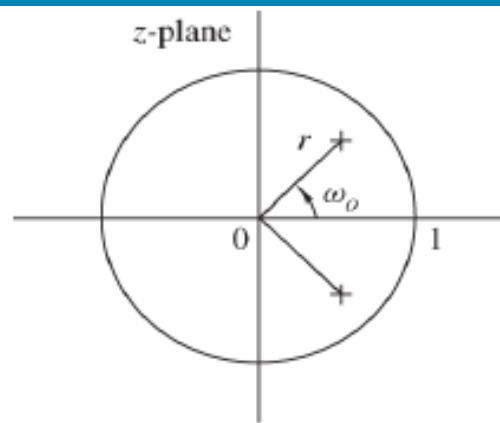
Two complex and conjugate poles and two zeros











# FIR

- FIR :

$$\left\{ \underset{\uparrow}{1}, 2, 0, 4 \right\} \quad h(n) = \sum_{j=0}^{N-1} b_j \delta(n-j)$$

- From a generic input  $x(n)$ , we obtain the output  $y(n)$ :

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{k=0}^3 h(k)x(n-k)$$

$$y(n) = x(n) + 2x(n-1) + 4x(n-3)$$

- Once  $h(n)$  is known the z-transform becomes:

$$H(z) = 1 + 2z^{-1} + 4z^{-3}$$



# Exercise

- Find the output  $y(n)$  of a system with:

$$H(z) = 1 - z^{-2} + 2z^{-3}$$

- When the input is  $x(n) = u(n)$ ;

$$h(n) = \delta(n) - \delta(n-2) + 2\delta(n-3)$$

$$y(n) = x(n) - x(n-2) + 2x(n-3)$$

# System stability

- The stability for a LTI system requires that the modulus of impulse response  $h(n)$  is *summable*

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

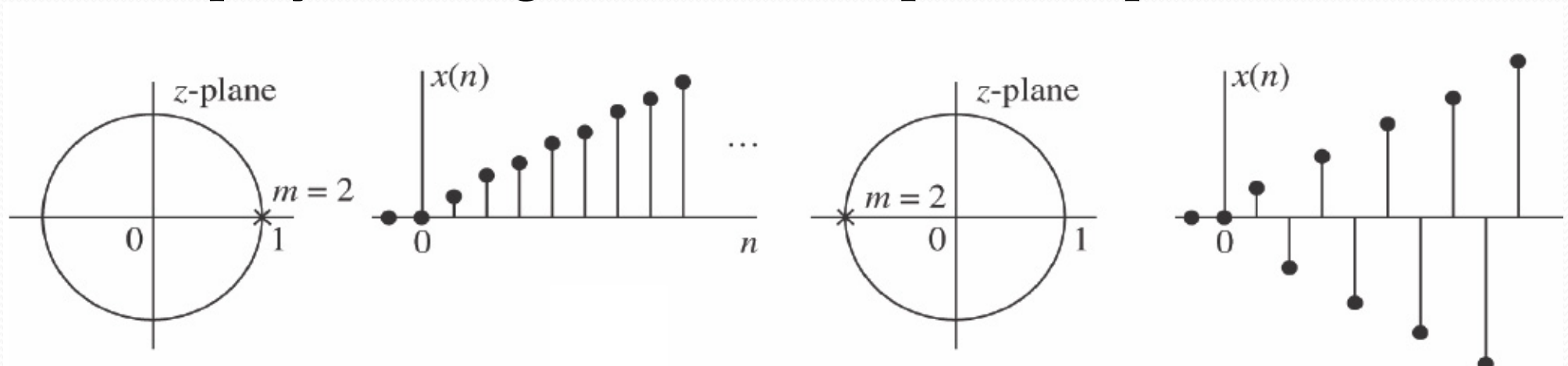
- If the system is causal the stability condition becomes: in the  $z$  domain the transfer function  $H(z)$  has all the poles inside the unitary circle of the  $z$ -plane.

$$H(z) = \frac{z}{z-3}$$



# Stability

- Multiple poles on the unitary circle induce a polynomial grow in the impulse response.



- Has a pole of order 2 in  $z=1$

the impulse response is

$h(n)=n u(n)$ : a slope function (it can be interpreted as the convolution of a step with itself).

$$H(z) = \frac{1}{z(z-1)^2}$$

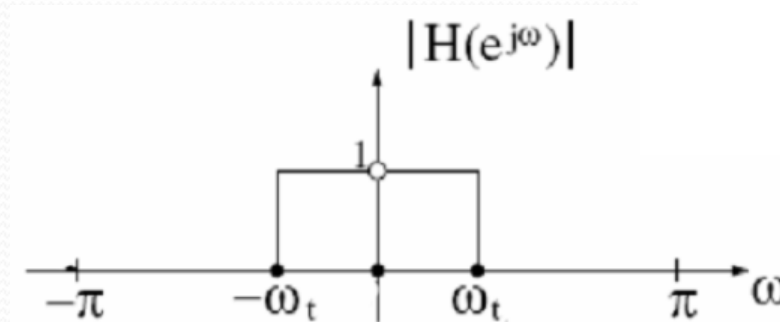
# Filter design by the positioning of poles and zeros

$$H(z) = \frac{N(z)}{D(z)} = Kz^{M-N} \frac{(z - c_1)(z - c_2) \dots (z - c_N)}{(z - d_1)(z - d_2) \dots (z - d_M)}$$

- The poles must be placed close to the unitary circle at the complex pulsations for the frequencies of the input signal  $x(n)$  that must be emphasised.
- Zeros must be placed closed to the unitary circle at the pulsations of the input signal  $x(n)$  *that must be attenuated*.

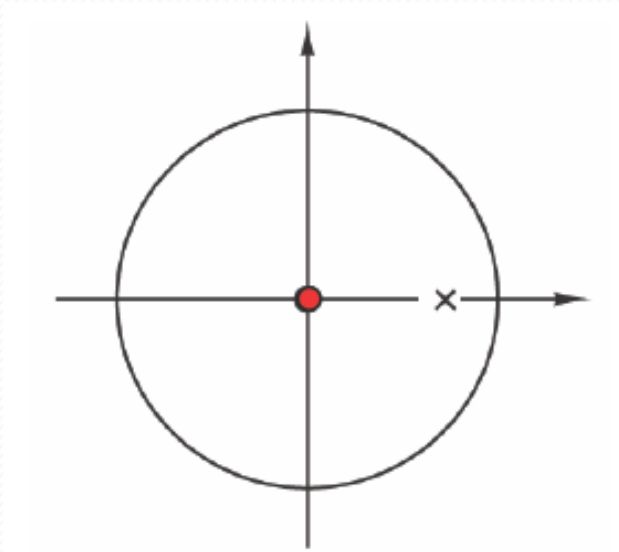


# Ideal low pass filter

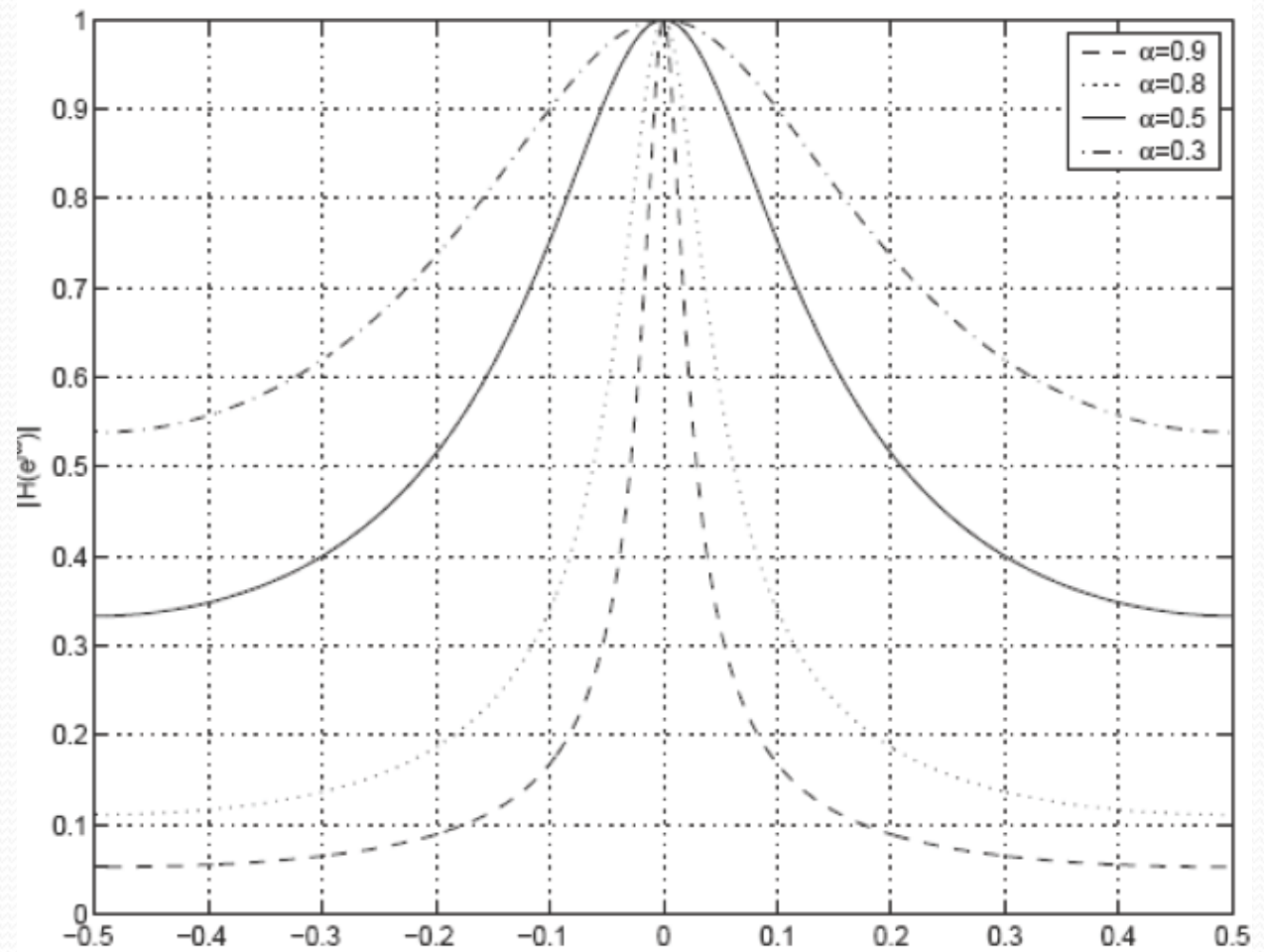


- Filter poles must be placed at pulsations of the pass band  $H(e^{j\omega}) : |\omega| \in [0, \omega_t]$
- Zeros must be placed on the unitary circle  $|z|=1$ , at the complementary pulsations  $|\omega| \in [\omega_t, \pi]$

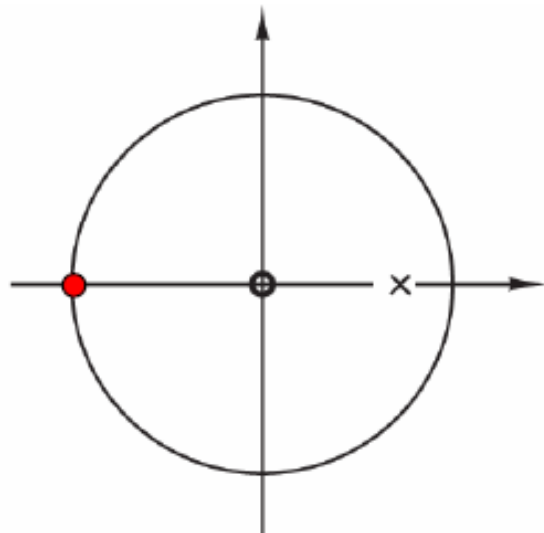
# Low pass filter 1 pole 1 zero



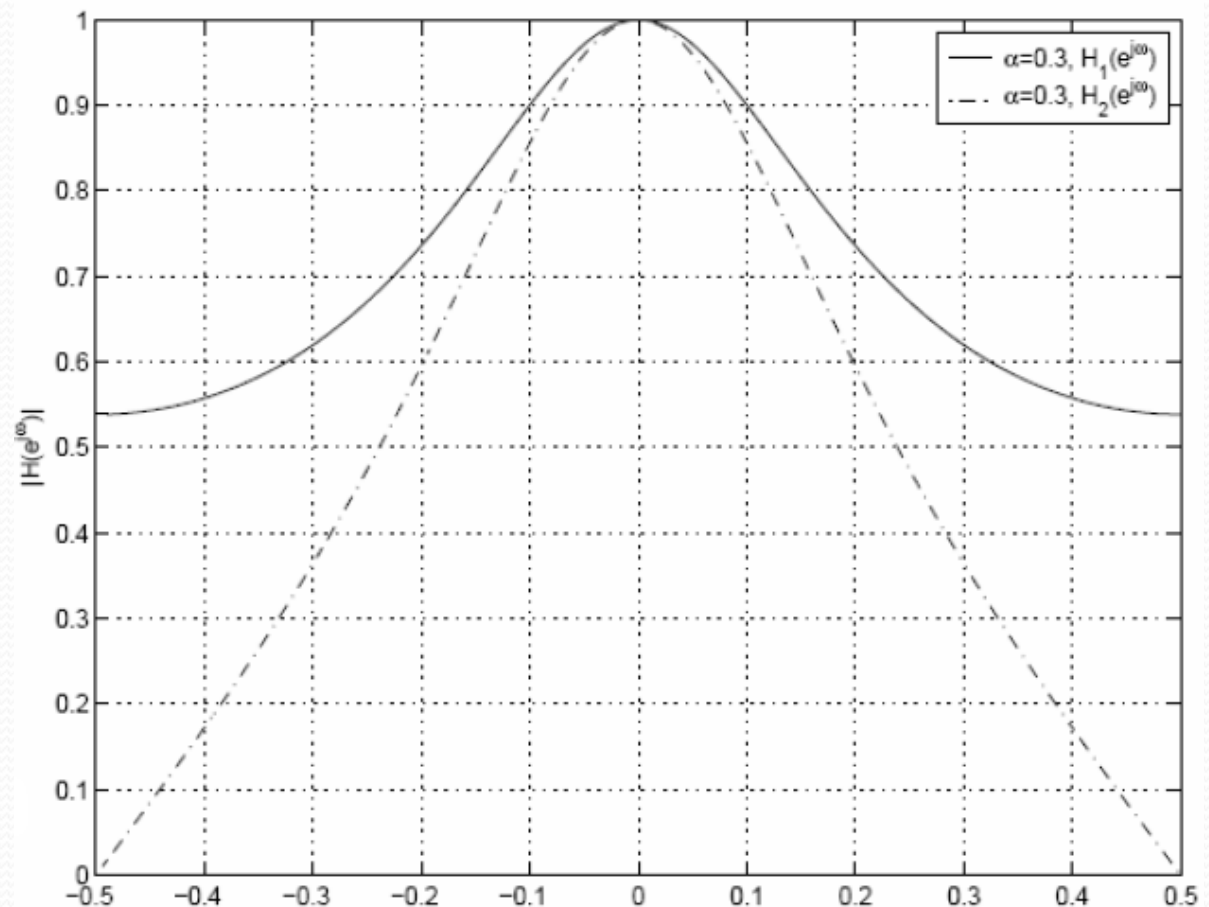
$$H_1(z) = \frac{1-\alpha}{1-\alpha z^{-1}} = z \frac{(1-\alpha)}{z-\alpha}$$



# Low pass filter 1 pole 1 zero



$$H_2(z) = \frac{1-\alpha}{2} \frac{(1+z^{-1})}{(1-\alpha z^{-1})} = \frac{1-\alpha}{2} \frac{(z+1)}{(z-\alpha)}$$

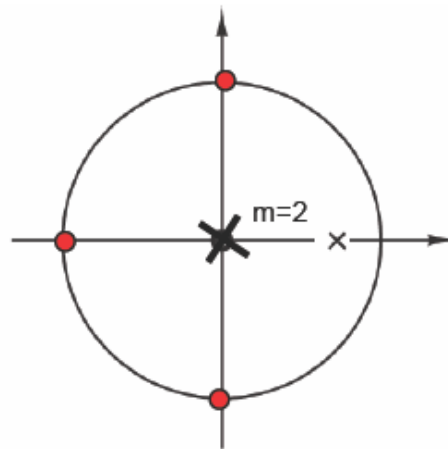




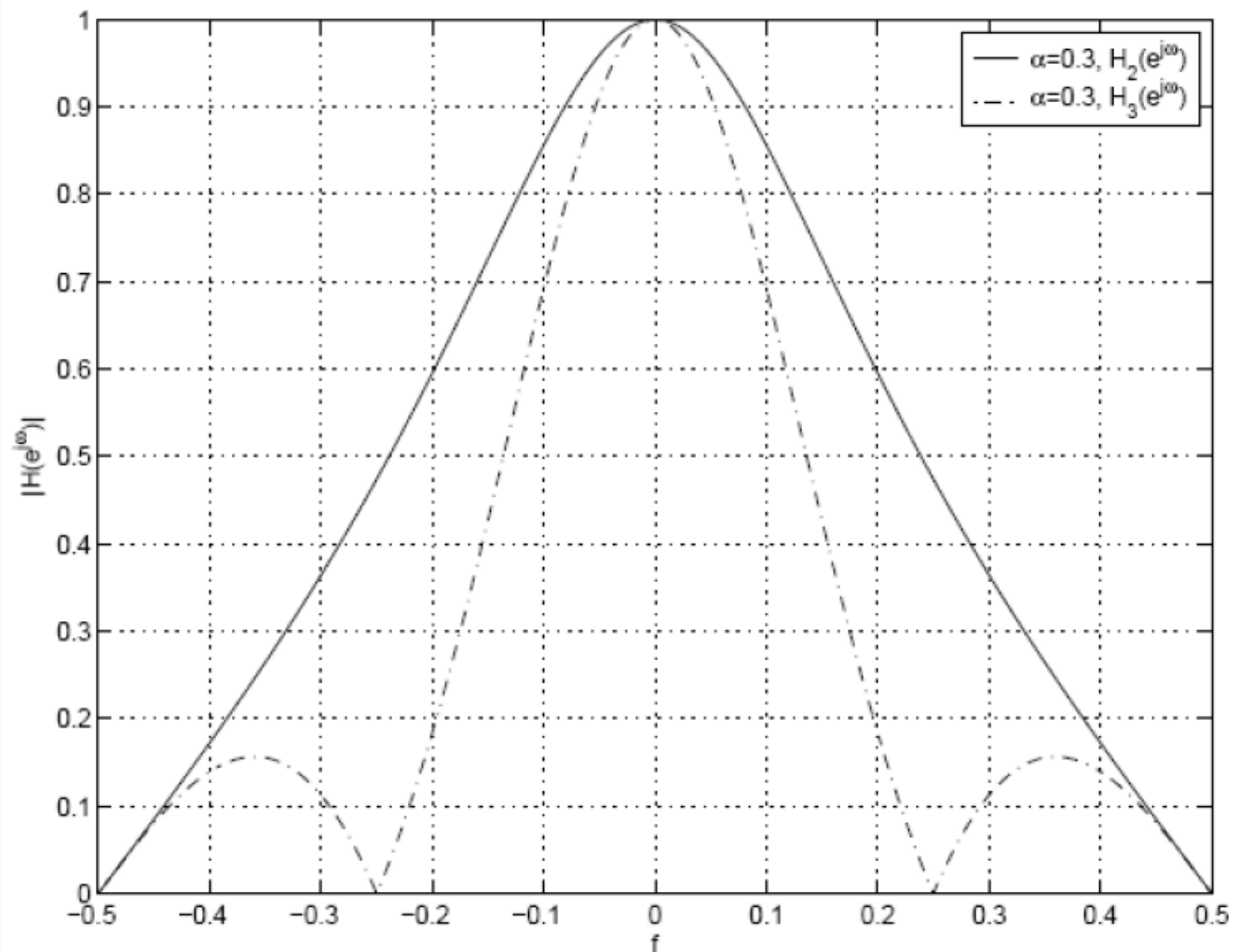
# Low pass filter with 3 zeros and 1 pole

- It is possible to emphasize the attenuation of the low pass filter at the high frequencies inserting further couples of complex and conjugate zeros ( for the physical realizability of a filter).

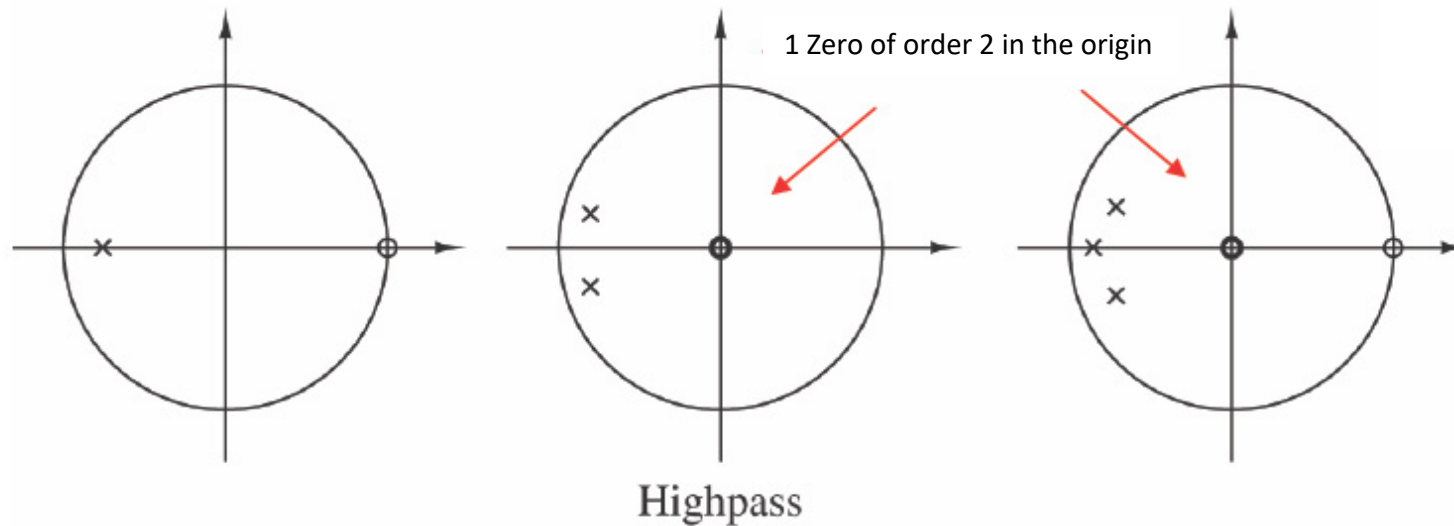
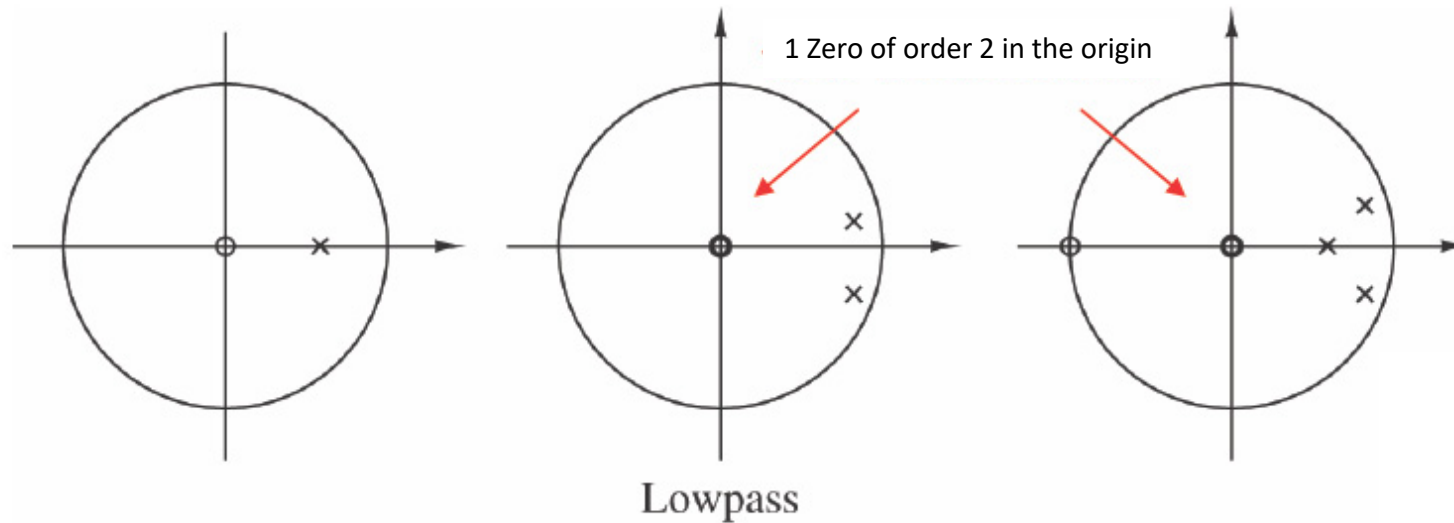
$$H_3(z) = \frac{1-\alpha}{4} \frac{(1+z^{-1})}{1-\alpha z^{-1}} (1-\beta z^{-1})(1-\beta^* z^{-1})$$



# Low pass filter with 3 zeros

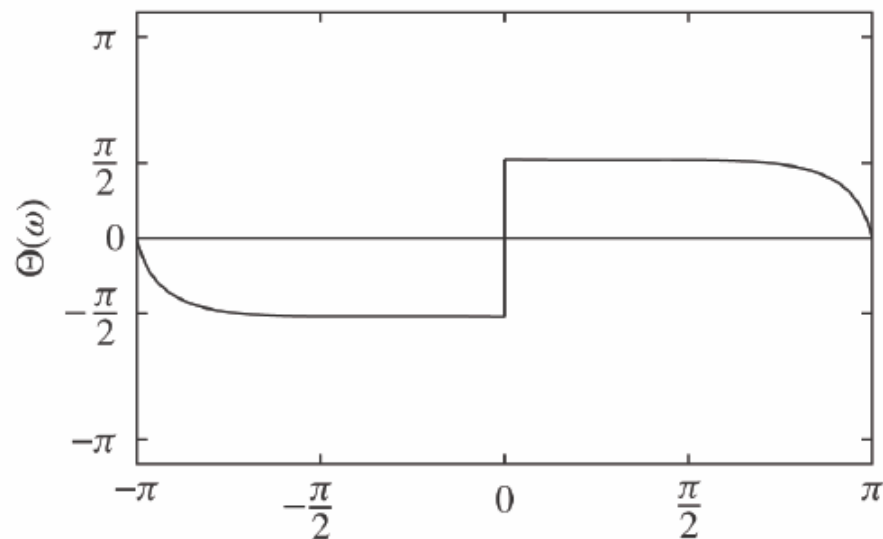
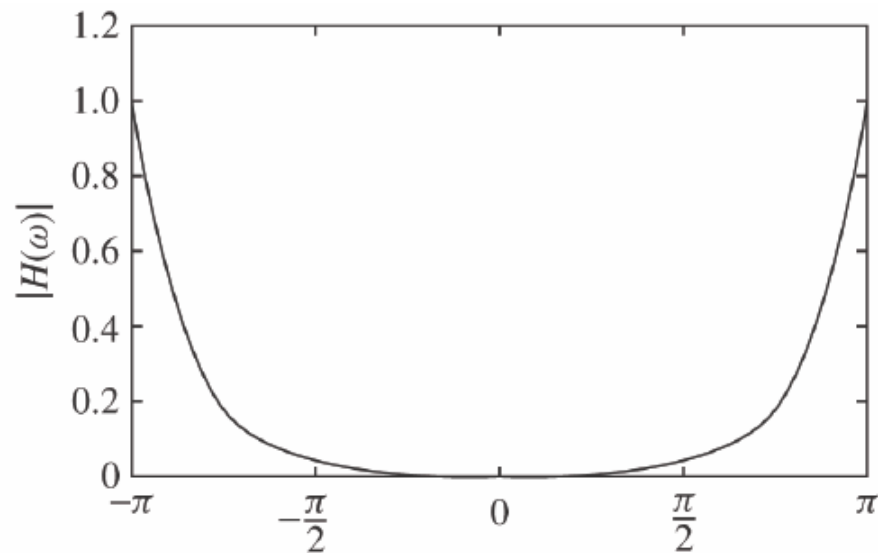


# Low pass and high pass bands

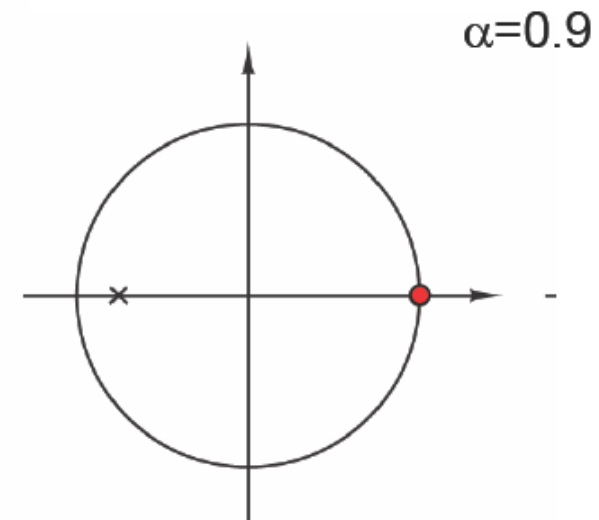




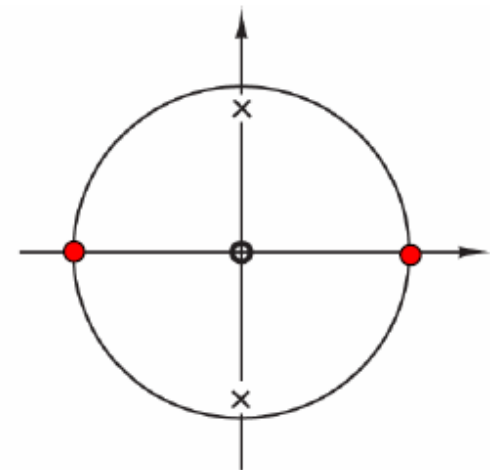
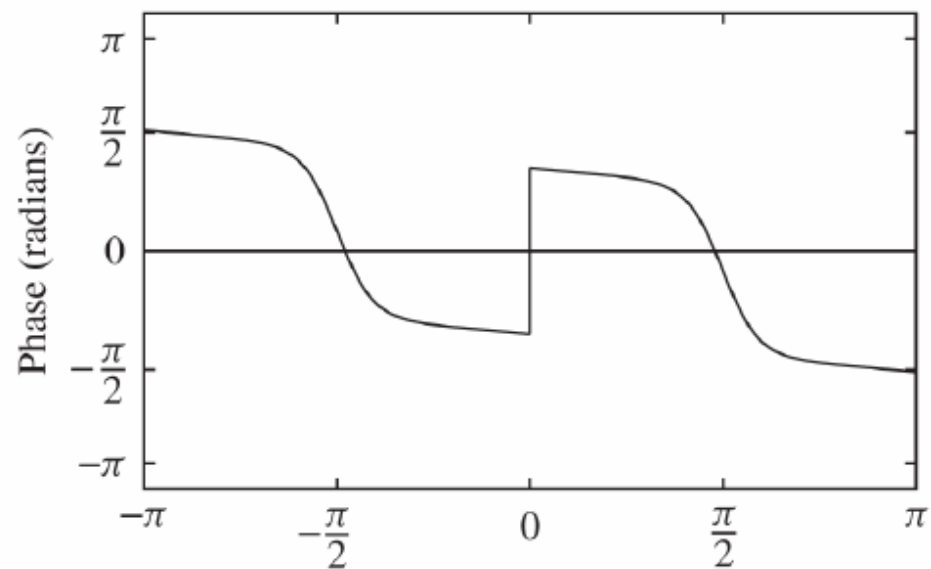
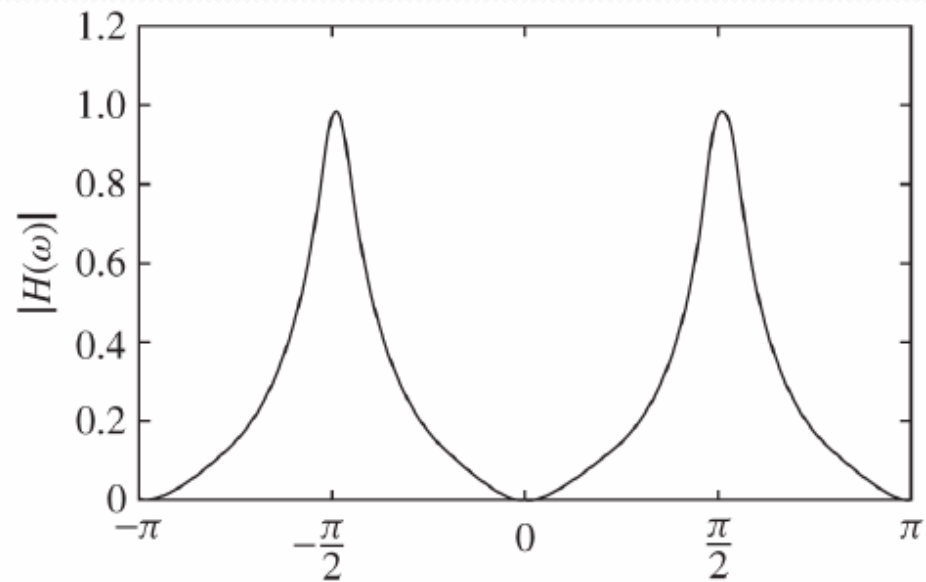
# High pass with one pole and one zero



$$H_2(z) = \frac{1-\alpha}{2} \frac{(1-z^{-1})}{(1+\alpha z^{-1})} = \frac{1-\alpha}{2} \frac{(z-1)}{(z+\alpha)}$$



# Band pass filter



# Amplitude and phase definition

- Two samples sequence: one zero system.

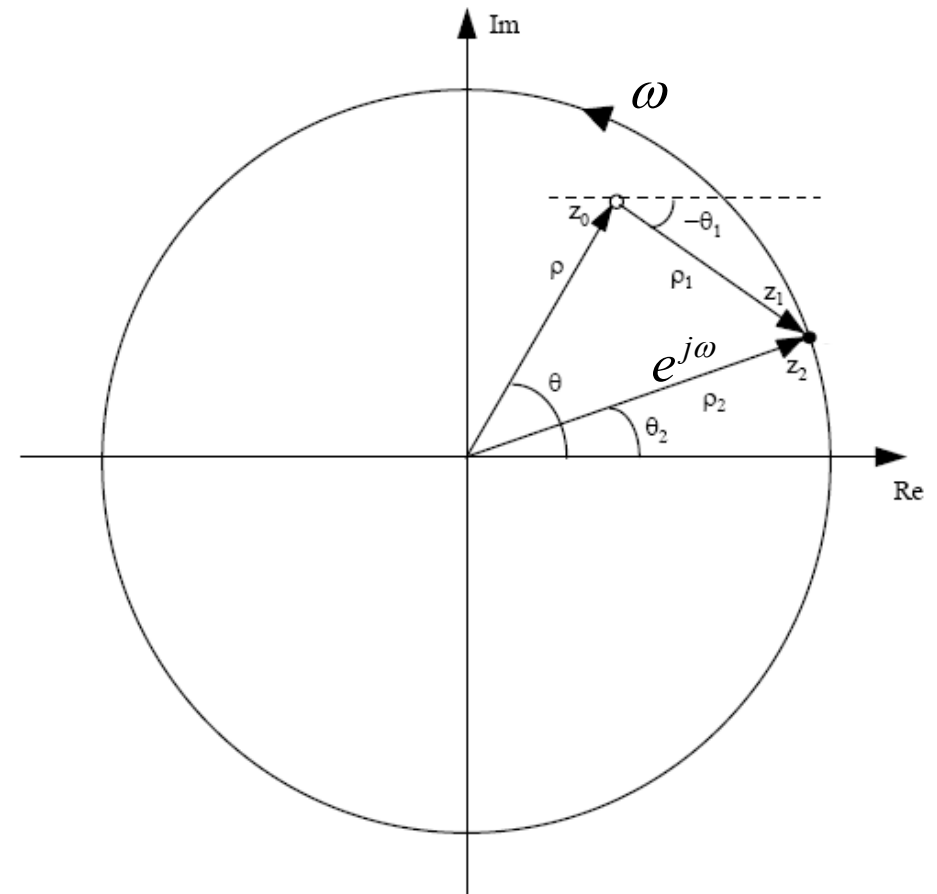
$$A(z) = 1 - z_0 z^{-1} = \frac{(z - z_0)}{z}$$

- Amplitude characteristic

$$|A(z)|_{z=e^{j\omega}} = \frac{|z - z_0|}{|z|} = |z - z_0|$$

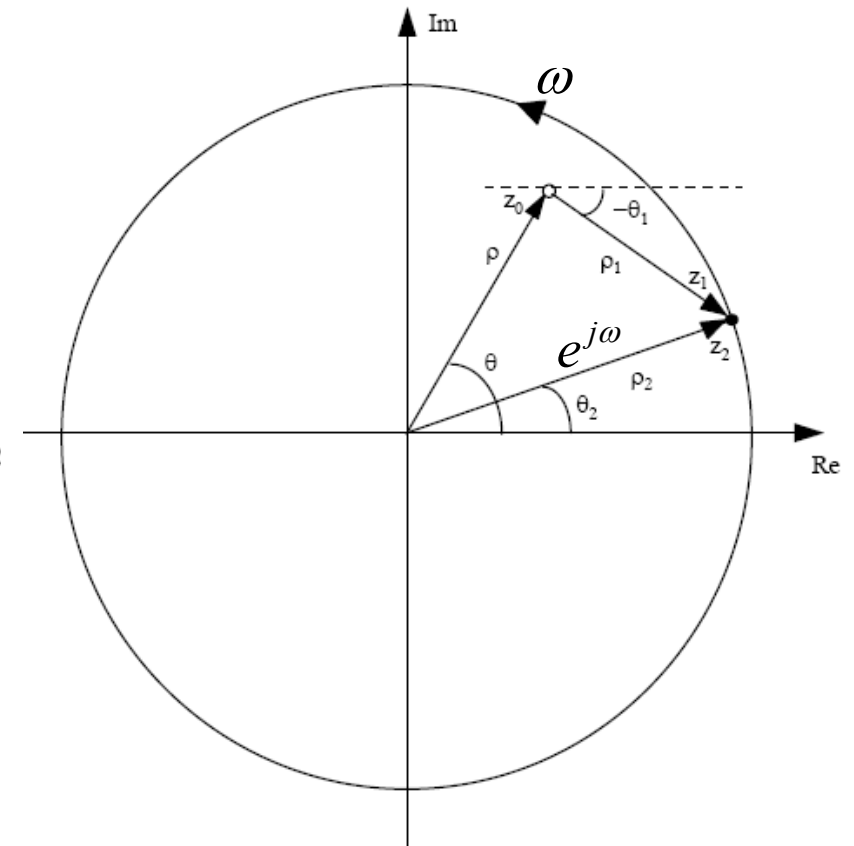
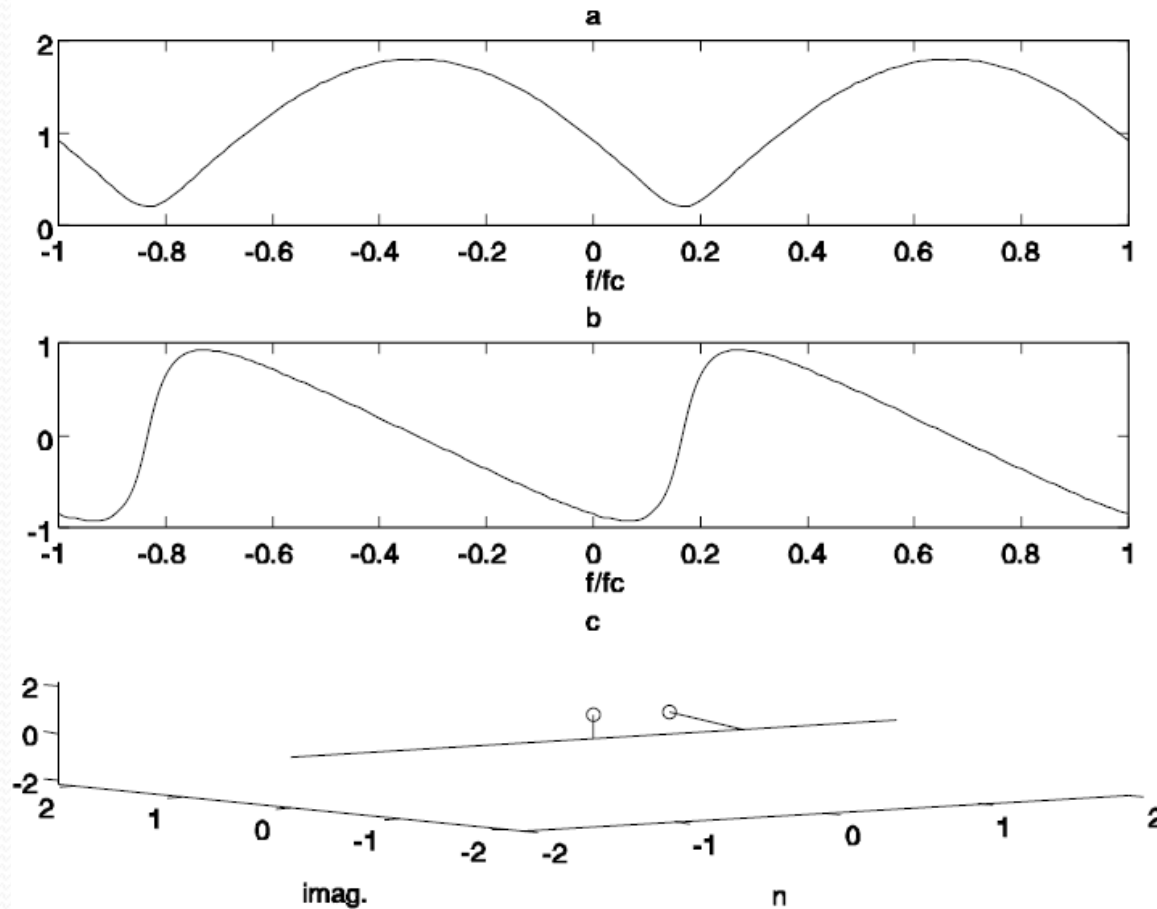
- Phase characteristic

$$\angle A(z)|_{z=e^{j\omega}} = \angle(z - z_0) - \angle(z)$$





# Amplitude and phase characteristic



# Zeros for maximum and minimum phase

- If zeros are on the unit circle they will delete true sinusoids.
- If they are outside they will delete generalized growing sinusoids.
- While, if they are inside they will delete generalized decreasing sinusoids.
- The zeros inside the unit circle are minimum phase zeros while the ones outside are called maximum phase zeros.



# Analysis for one zero

$$A(z) = 1 - z^{-1}$$

- The phase behavior in the origin presents a discontinuity:

$$\begin{aligned}\angle \left( A(z) \Big|_{z=e^{j\omega}} \right) &= \angle \left( 1 - e^{-j\omega} \right) = \\ &= \angle \left( e^{-j\omega/2} \left( e^{j\omega/2} - e^{-j\omega/2} \right) \cdot \frac{2j}{2j} \right) = \angle \left( 2je^{-j\omega/2} \sin \frac{\omega}{2} \right) = \\ &= \angle \left( e^{j\pi/2} e^{-j\omega/2} \sin \frac{\omega}{2} \right) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & \text{for } \sin \frac{\omega}{2} \geq 0 \text{ i.e. } \omega \geq 0 \\ \frac{\pi}{2} - \frac{\omega}{2} - \pi = -\frac{\pi}{2} - \frac{\omega}{2} & \text{for } \sin \frac{\omega}{2} < 0 \text{ i.e. } \omega < 0 \end{cases}\end{aligned}$$

- While, if we consider a zero not on the unit circle we will obtain:

$$\angle \left( 1 - \rho z^{-1} \Big|_{z=e^{j\omega}} \right)$$

- If the zero is inside the unit circle the phase characteristic for the zero frequency is null and continuous.



# Reciprocal and conjugate zeros

- When a zero is really close to the unit circle minimal variations of the sequence samples can generate high variations in the phase characteristic.
- It is interesting to notice that two zeros reciprocal and conjugate zeros (i.e. two sequences, each made of 2 samples, whose z-transforms present one zero in  $z_0$  or one zero in  $1/z_0^*$ ) present the same amplitude response (apart from a constant coefficient) while have completely different phase characteristic.

$$|A(z)|^2 = |1 - z_0 z^{-1}|^2 = |1 - z_0 e^{-j\omega}|^2 = 1 + |z_0|^2 - 2 \operatorname{Re}(z_0 e^{-j\omega})$$

$$|A_r(z)|^2 = \left| 1 - \frac{1}{z_0^*} z^{-1} \right|^2 = \frac{|z_0^* - e^{-j\omega}|^2}{|z_0|^2} = \frac{1 + |z_0|^2 - 2 \operatorname{Re}(z_0 e^{-j\omega})}{|z_0|^2} = \frac{|A(z)|^2}{|z_0|^2}$$

# The inverse z-transform

Formally, the inverse z-transform can be performed by evaluating a Cauchy integral. However, for discrete LTI systems simpler methods are often sufficient.

## 3.1 Inspection method

If one is familiar with (or has a table of) common z-transform pairs, the inverse can be found by inspection. For example, one can invert the z-transform

$$X(z) = \left( \frac{1}{1 - \frac{1}{2}z^{-1}} \right), \quad |z| > \frac{1}{2},$$

using the z-transform pair

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad \text{for } |z| > |a|.$$

# Inspection method (cont.)

By inspection we recognise that

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

Also, if  $X(z)$  is a sum of terms then one may be able to do a term-by-term inversion by inspection, yielding  $x[n]$  as a sum of terms.



# Partial fraction expansion

For any rational function we can obtain a partial fraction expansion, and identify the z-transform of each term. Assume that  $X(z)$  is expressed as a ratio of polynomials in  $z^{-1}$ :

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}.$$

It is always possible to factor  $X(z)$  as

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})},$$

where the  $c_k$ 's and  $d_k$ 's are the nonzero zeros and poles of  $X(z)$ .

# Partial fraction expansion

- If  $M < N$  and the poles are all first order, then  $X(z)$  can be expressed as

$$X(z) = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}.$$

In this case the coefficients  $A_k$  are given by

$$A_k = (1 - d_k z^{-1})X(z) \Big|_{z=d_k}.$$

- If  $M \geq N$  and the poles are all first order, then an expansion of the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}$$

can be used, and the  $B_r$ 's be obtained by long division of the numerator by the denominator. The  $A_k$ 's can be obtained using the same equation as for  $M < N$ .

# Power series expansion

If the z-transform is given as a power series in the form

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \dots + x[-2]z^2 + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots, \end{aligned}$$

then any value in the sequence can be found by identifying the coefficient of the appropriate power of  $z^{-1}$ .



# Finite length sequence

## Example: finite-length sequence

The z-transform

$$X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$$

can be multiplied out to give

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

By inspection, the corresponding sequence is therefore

$$x[n] = \begin{cases} 1 & n = -2 \\ -\frac{1}{2} & n = -1 \\ -1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$x[n] = 1\delta[n + 2] - \frac{1}{2}\delta[n + 1] - 1\delta[n] + \frac{1}{2}\delta[n - 1].$$

# Power series expansion

Consider the transform

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|.$$

Since the ROC is the exterior of a circle, the sequence is right-sided. We therefore divide to get a power series in powers of  $z^{-1}$ :

$$\begin{array}{r} 1 + az^{-1} + a^2z^{-2} + \dots \\ 1 - az^{-1} \overline{) 1} \\ \underline{1 - az^{-1}} \phantom{+ \dots} \\ az^{-1} - a^2z^{-2} \phantom{+ \dots} \\ \underline{az^{-1} - a^2z^{-2}} \phantom{+ \dots} \\ a^2z^{-2} + \dots \end{array}$$

or

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \dots$$

# Properties of the z-transform

- Linearity

The linearity property is as follows:

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z), \quad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$



# Time shifting

The time-shifting property is as follows:

$$x[n - n_0] \xleftrightarrow{Z} z^{-n_0} X(z), \quad \text{ROC} = R_x.$$

(The ROC may change by the possible addition or deletion of  $z = 0$  or  $z = \infty$ .) This is easily shown:

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)} \\ &= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z). \end{aligned}$$

# Example: shifted exponential sequence

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \quad |z| > \frac{1}{4}.$$

From the ROC, this is a right-sided sequence. Rewriting,

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = z^{-1} \left( \frac{1}{1 - \frac{1}{4}z^{-1}} \right), \quad |z| > \frac{1}{4}.$$

The term in brackets corresponds to an exponential sequence  $(1/4)^n u[n]$ . The factor  $z^{-1}$  shifts this sequence one sample to the right. The inverse z-transform is therefore

$$x[n] = (1/4)^{n-1} u[n-1].$$

Note that this result could also have been easily obtained using a partial fraction expansion.

# Multiplication by an exponential sequence

The exponential multiplication property is

$$z_0^n x[n] \xleftrightarrow{Z} X(z/z_0), \quad \text{ROC} = |z_0| R_x,$$

where the notation  $|z_0| R_x$  indicates that the ROC is scaled by  $|z_0|$  (that is, inner and outer radii of the ROC scale by  $|z_0|$ ). All pole-zero locations are similarly scaled by a factor  $z_0$ : if  $X(z)$  had a pole at  $z = z_1$ , then  $X(z/z_0)$  will have a pole at  $z = z_0 z_1$ .

- If  $z_0$  is positive and real, this operation can be interpreted as a shrinking or expanding of the  $z$ -plane — poles and zeros change along radial lines in the  $z$ -plane.
- If  $z_0$  is complex with unit magnitude ( $z_0 = e^{j\omega_0}$ ) then the scaling operation corresponds to a rotation in the  $z$ -plane by an angle  $\omega_0$ . That is, the poles and zeros rotate along circles centered on the origin. This can be interpreted as a shift in the frequency domain, associated with modulation in the time domain by  $e^{j\omega_0 n}$ . If the Fourier transform exists, this becomes

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)}).$$



# Exponential multiplication

The z-transform pair

$$u[n] \xleftrightarrow{z} \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

can be used to determine the z-transform of  $x[n] = r^n \cos(\omega_0 n) u[n]$ . Since  $\cos(\omega_0 n) = 1/2 e^{j\omega_0 n} + 1/2 e^{-j\omega_0 n}$ , the signal can be rewritten as

$$x[n] = \frac{1}{2} (r e^{j\omega_0})^n u[n] + \frac{1}{2} (r e^{-j\omega_0})^n u[n].$$

# Exponential multiplication

From the exponential multiplication property,

$$\frac{1}{2}(re^{j\omega_0})^n u[n] \xleftrightarrow{z} \frac{1/2}{1 - re^{j\omega_0} z^{-1}}, \quad |z| > r$$

$$\frac{1}{2}(re^{-j\omega_0})^n u[n] \xleftrightarrow{z} \frac{1/2}{1 - re^{-j\omega_0} z^{-1}}, \quad |z| > r,$$

so

$$\begin{aligned} X(z) &= \frac{1/2}{1 - re^{j\omega_0} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_0} z^{-1}}, \quad |z| > r \\ &= \frac{1 - r \cos \omega_0 z^{-1}}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}}, \quad |z| > r. \end{aligned}$$

# Differentiation

The differentiation property states that

$$nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}, \quad \text{ROC} = R_x.$$

This can be seen as follows: since

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

we have

$$-z \frac{dX(z)}{dz} = -z \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$



# Evaluating convolution by z-trans.

The z-transforms of the signals  $x_1[n] = a^n u[n]$  and  $x_2[n] = u[n]$  are

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}, \quad |z| > 1.$$

For  $|a| < 1$ , the z-transform of the convolution  $y[n] = x_1[n] * x_2[n]$  is

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \quad |z| > 1.$$

Using a partial fraction expansion,

$$Y(z) = \frac{1}{1 - a} \left( \frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}} \right), \quad |z| > 1,$$

so

$$y[n] = \frac{1}{1 - a} (u[n] - a^{n+1} u[n]).$$

# Common z-transform pairs

Sequence	Transform	ROC
$\delta[n]$	1	All $z$
$u[n]$	$\frac{1}{1-z^{-1}}$	$ z  > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z  < 1$
$\delta[n-m]$	$z^{-m}$	All $z$ except 0 or $\infty$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z  >  a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  >  a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  <  a $
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z  > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z  > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$ z  > r$

# Relationship with Laplace transform

Continuous-time systems and signals are usually described by the Laplace transform. Letting  $z = e^{sT}$ , where  $s$  is the complex Laplace variable

$$s = d + j\omega,$$

we have

$$z = e^{(d+j\omega)T} = e^{dT} e^{j\omega T}.$$

Therefore

$$|z| = e^{dT} \quad \text{and} \quad \angle z = \omega T = 2\pi f / f_s = 2\pi \omega / \omega_s,$$

where  $\omega_s$  is the sampling frequency. As  $\omega$  varies from  $-\infty$  to  $\infty$ , the s-plane is mapped to the z-plane:

- The  $j\omega$  axis in the s-plane is mapped to the unit circle in the z-plane.
- The left-hand s-plane is mapped to the inside of the unit circle.
- The right-hand s-plane maps to the outside of the unit circle.



# Examples and properties

- Complex conjugates zeros or poles:

$$1 - 2\rho \cos(\theta) z^{-1} + \rho^2 z^{-2} = 0 \rightarrow z_{1,2} = \rho(\cos(\theta) \pm j \sin(\theta))$$

- An infinite number of zeros approximates a (stable) pole:

$$\begin{aligned} \frac{1}{1 - az^{-1}} &= \sum_{n=0}^{\infty} (az^{-1})^n = \lim_{m \rightarrow +\infty} \sum_{n=0}^m (az^{-1})^n = \\ &= \lim_{m \rightarrow +\infty} \frac{1 - a^{m+1} z^{-(m+1)}}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \dots \end{aligned}$$

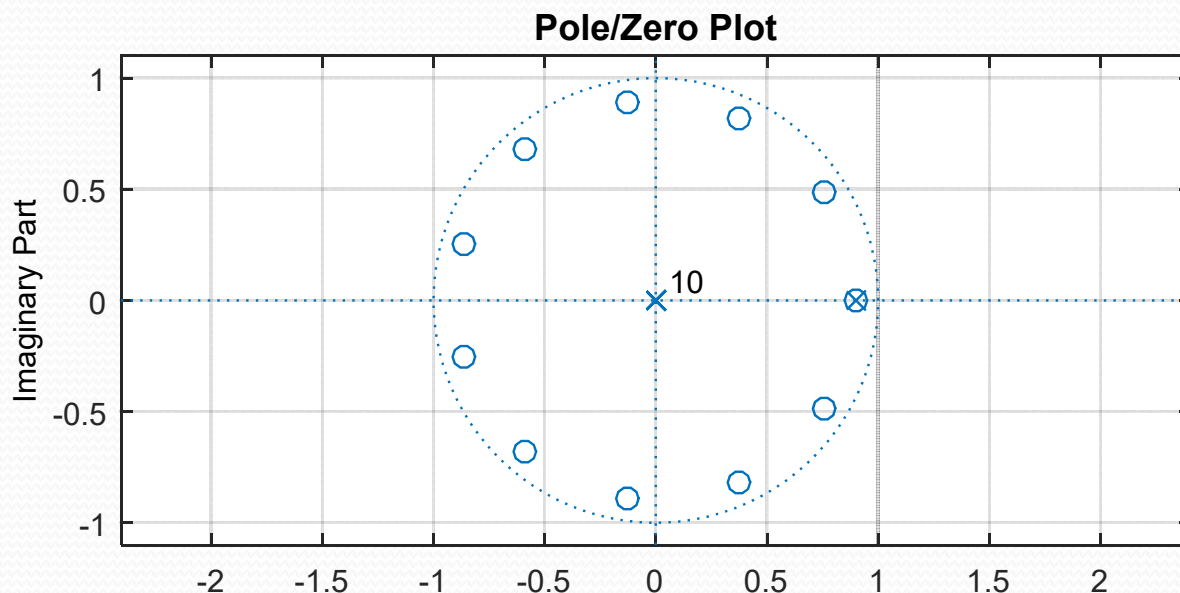
# Examples and properties

$$\frac{1}{1 - az^{-1}} = \lim_{m \rightarrow +\infty} \frac{1 - a^{m+1} z^{-(m+1)}}{1 - az^{-1}} \approx \frac{1 - a^{\bar{m}+1} z^{-(\bar{m}+1)}}{1 - az^{-1}} \Bigg|_{m=\bar{m}}$$

$$roots : \begin{cases} num : & z_{1,\bar{m}+1} = \sqrt[\bar{m}+1]{a^{\bar{m}+1}} \\ den : & z_1 = a \end{cases}$$

The pole is deleted by a zero in the same position

$$H(z) = \frac{1 - 0.9^{11} z^{-11}}{1 - 0.9z^{-1}}$$





# All pass filters

- An LTI with a zero and a pole in reciprocal conjugate position is the simplest All Pass Filter:

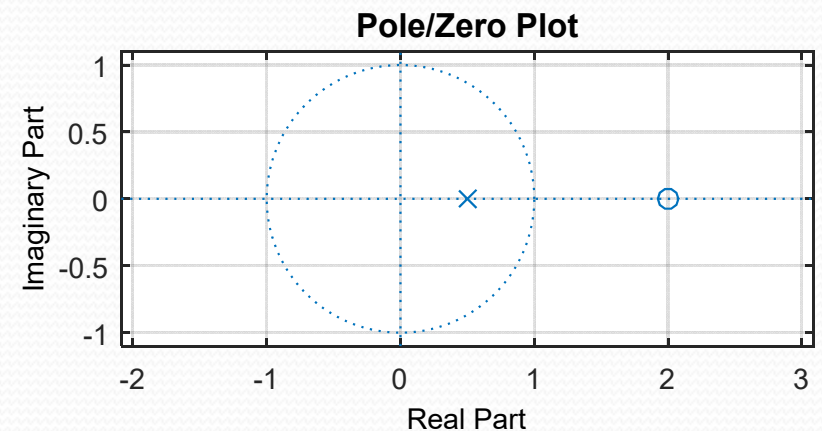
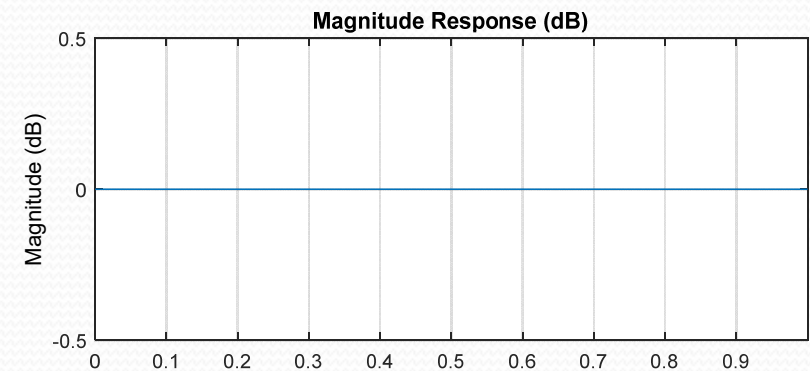
$$H(z) = \frac{c + z^{-1}}{1 + cz^{-1}}$$

$$c = \frac{\tan(\pi f_c / f_s) - 1}{\tan(\pi f_c / f_s) + 1}$$

$f_c$  = cut-off frequency

$f_s$  = sampling frequency

- In this case  $c = -\frac{1}{2}$





# All pass filters

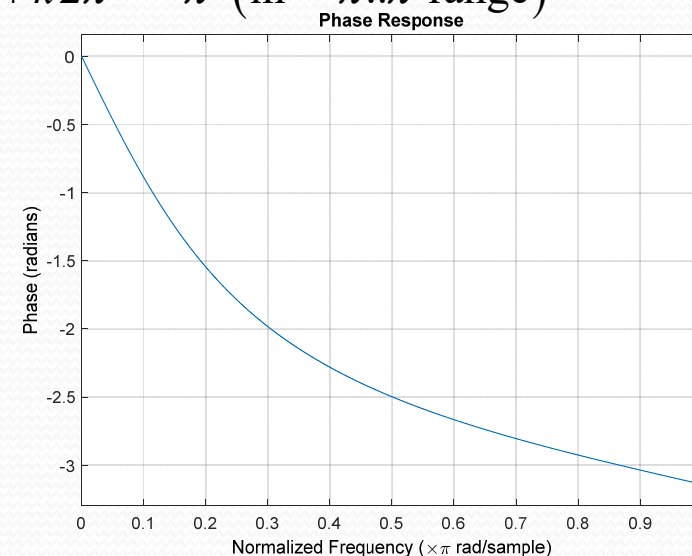
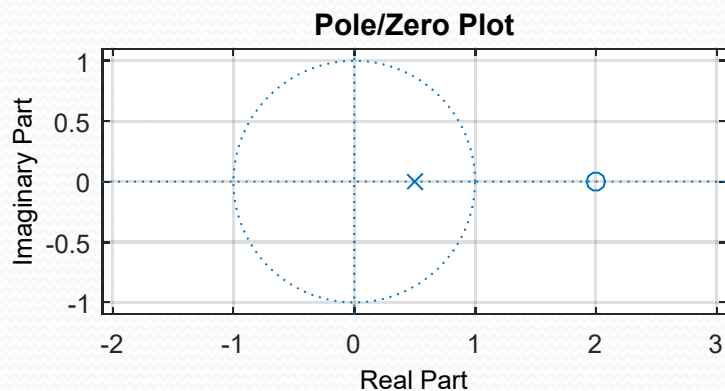
$$H(z) = \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2}z^{-1}} = -\frac{\frac{1}{2} - z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\angle H(z)\big|_{z=e^{j\omega}} = \angle(-1) + \angle(z-2)\big|_{z=e^{j\omega}} - \angle\left(z - \frac{1}{2}\right)\bigg|_{z=e^{j\omega}} = \pi + \operatorname{atan} 2(\sin \omega, \cos \omega - 2) - \operatorname{atan} 2\left(\sin \omega, \cos \omega - \frac{1}{2}\right) + k2\pi$$

$$\angle H(\omega=0) = \pi + \operatorname{atan} 2(0, 1-2) - \operatorname{atan} 2\left(0, 1-\frac{1}{2}\right) = \pi + \pi - 0 + k2\pi = 0 \quad (\text{in } -\pi.. \pi \text{ range})$$

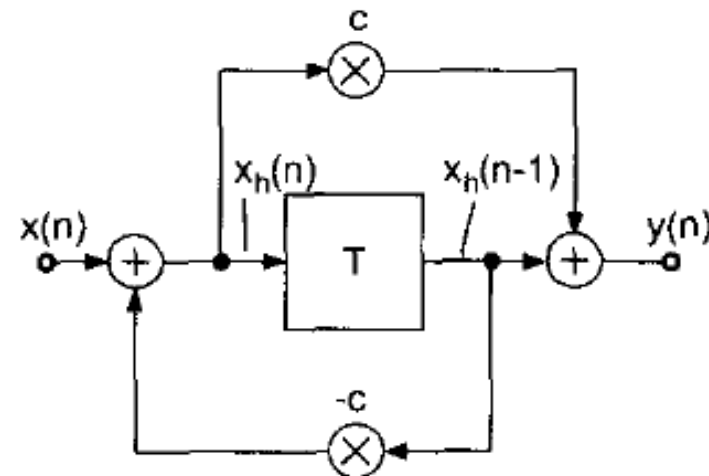
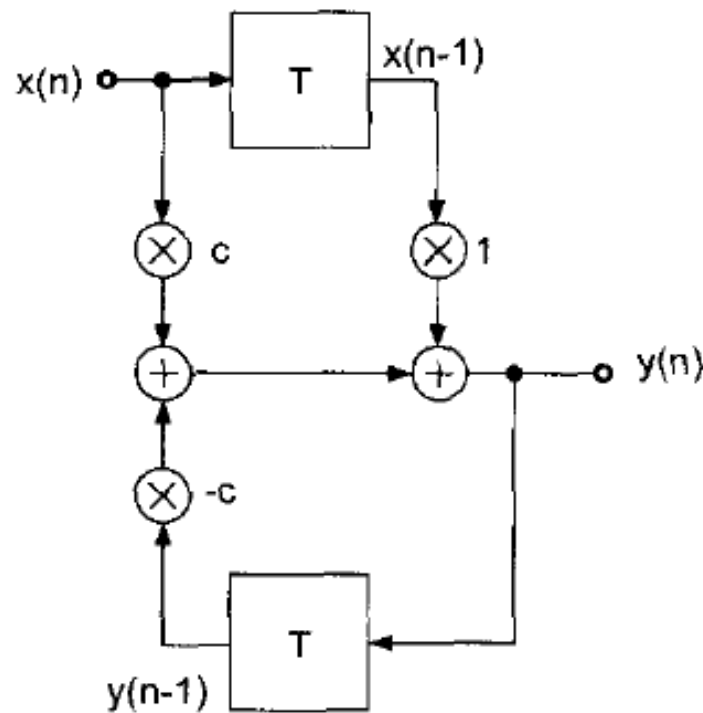
$$\angle H\left(\omega = \frac{\pi}{2}\right) = \pi + \operatorname{atan} 2(1, -2) - \operatorname{atan} 2\left(1, -\frac{1}{2}\right) \approx \pi + 2.68 - 2.03 + k2\pi \approx -2.49 \quad (\text{in } -\pi.. \pi \text{ range})$$

$$\angle H(\omega = \pi) = \pi + \operatorname{atan} 2(0, -1-2) - \operatorname{atan} 2\left(0, -1-\frac{1}{2}\right) = \pi + \pi - \pi + k2\pi = -\pi \quad (\text{in } -\pi.. \pi \text{ range})$$



# All pass filter implementation

$$y(n) = cx(n) + x(n-1) - cy(n-1),$$



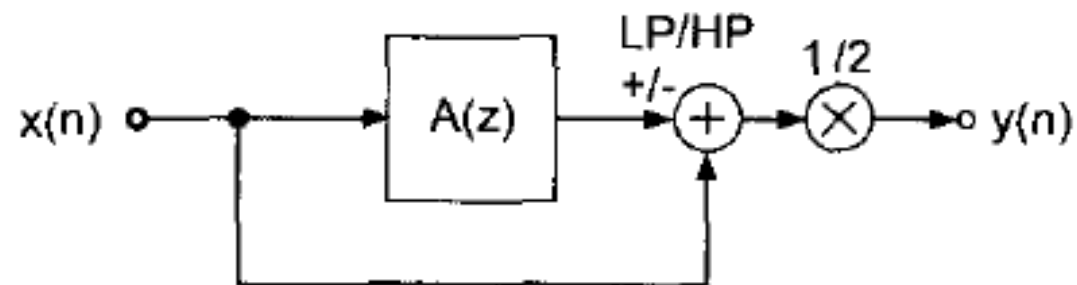
$$\begin{aligned} x_h(n) &= x(n) - cx_h(n-1) \\ y(n) &= cx_h(n) + x_h(n-1). \end{aligned}$$

# First order low/high pass

$$H(z) = \frac{1}{2} (1 \pm A(z)) \quad (\text{LP/HP } +/-)$$

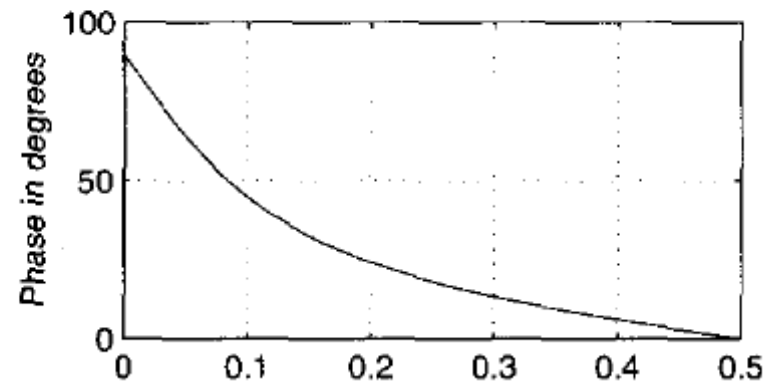
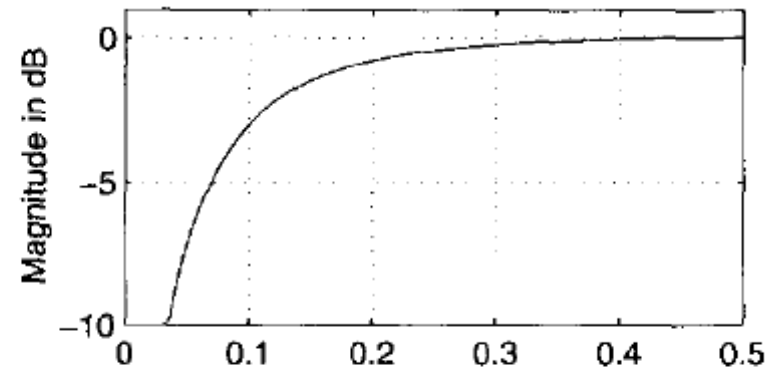
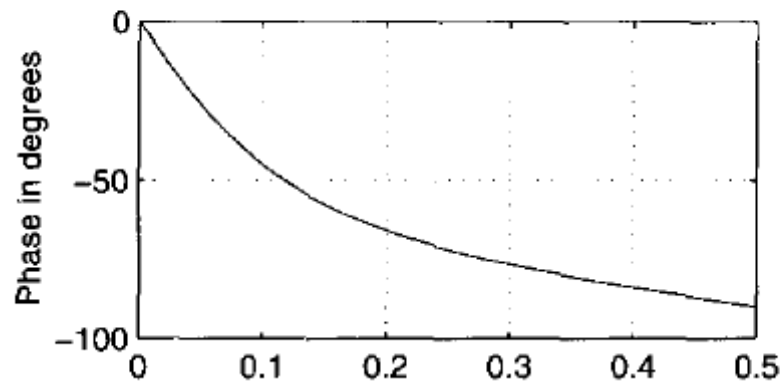
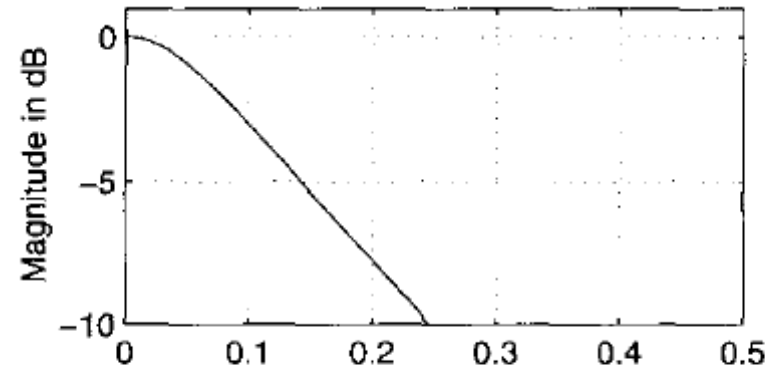
$$A(z) = \frac{z^{-1} + c}{1 + cz^{-1}}$$

$$c = \frac{\tan(\pi f_c/f_s) - 1}{\tan(\pi f_c/f_s) + 1}$$





# First order low/high pass



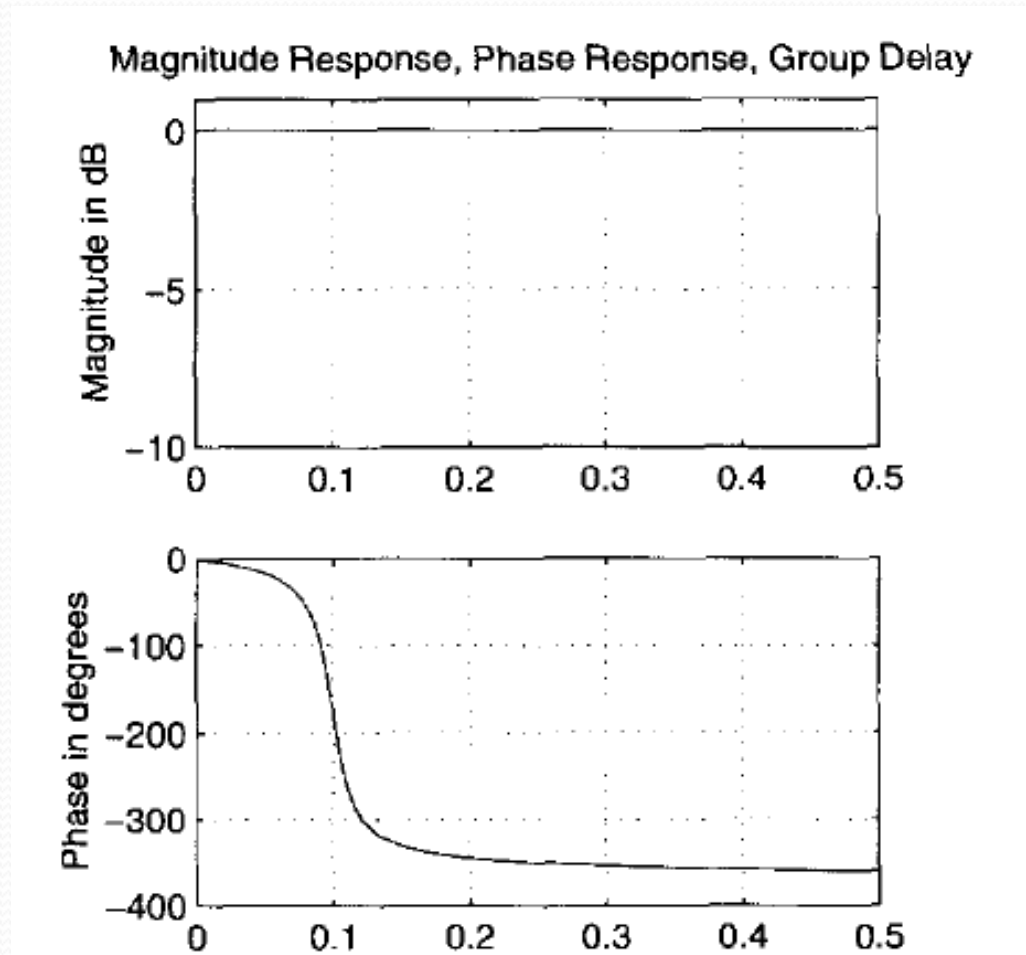
# Second order all-pass filter

$$\begin{aligned}A(z) &= \frac{-c + d(1 - c)z^{-1} + z^{-2}}{1 + d(1 - c)z^{-1} - cz^{-2}} \\c &= \frac{\tan(\pi f_b / f_s) - 1}{\tan(\pi f_b / f_s) + 1} \\d &= -\cos(2\pi f_c / f_s).\end{aligned}$$

- $d$  adjusts the cut-off frequency while  $c$  the bandwidth.

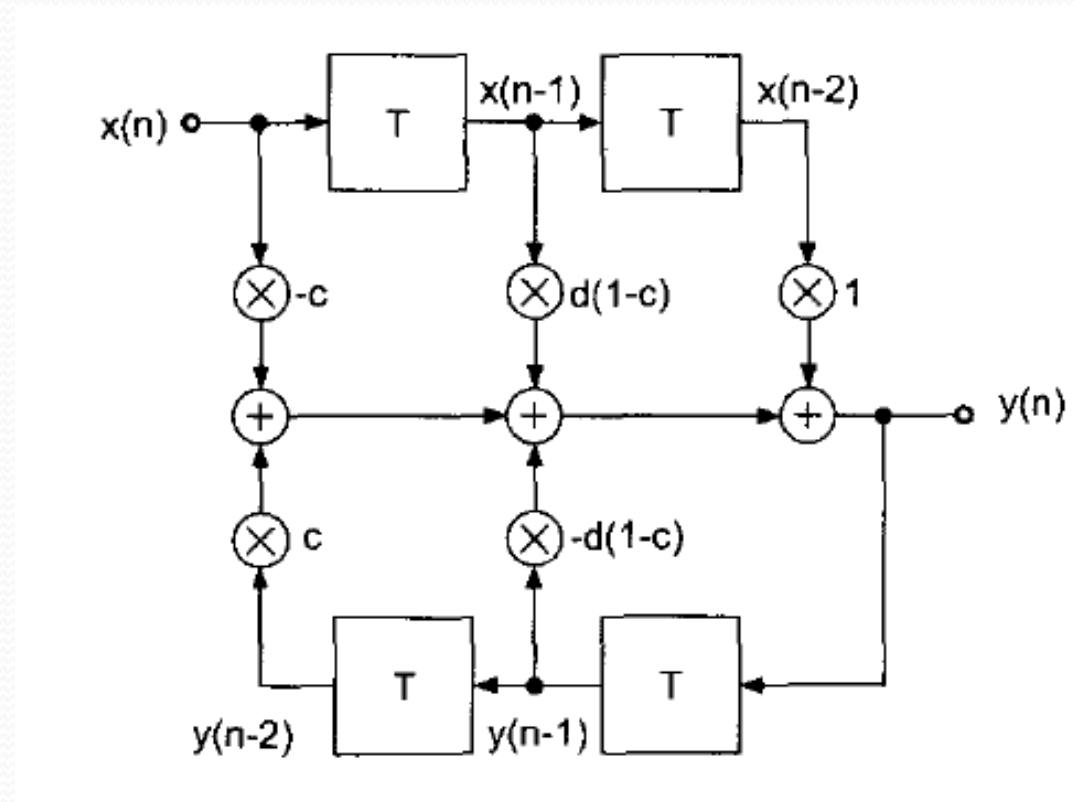
$$\begin{aligned}y(n) &= -cx(n) + d(1 - c)x(n - 1) + x(n - 2) \\&\quad - d(1 - c)y(n - 1) + cy(n - 2)\end{aligned}$$

# Second order all-pass filter





# Implementation of a second order all pass filter



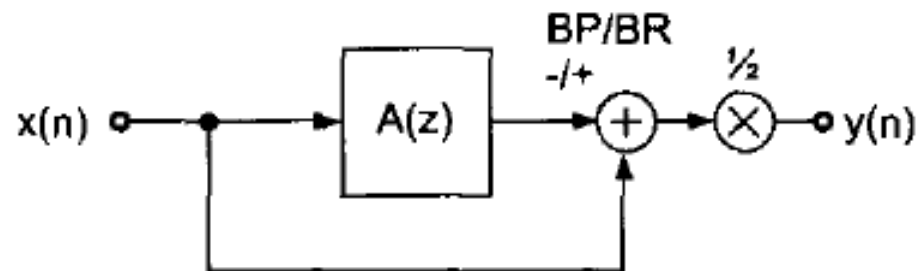
# Second order band pass/band reject

$$H(z) = \frac{1}{2} [1 \mp A(z)] \quad (\text{BP/BR } -/+)$$

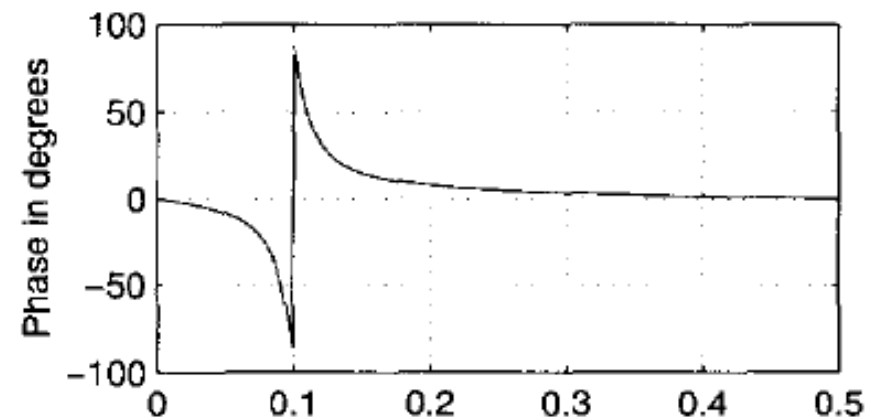
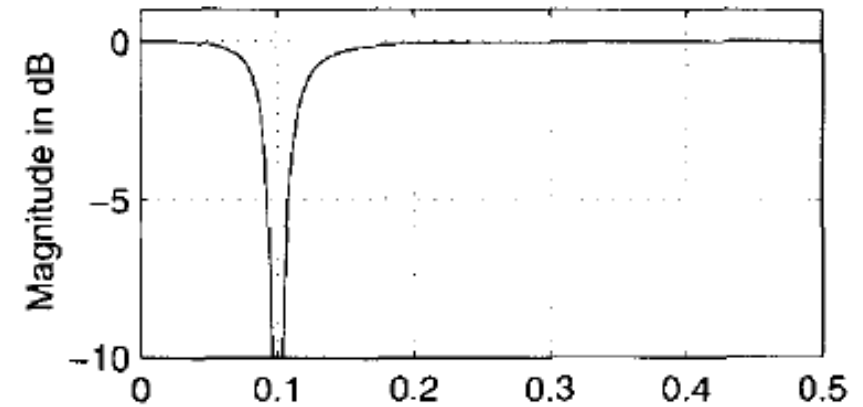
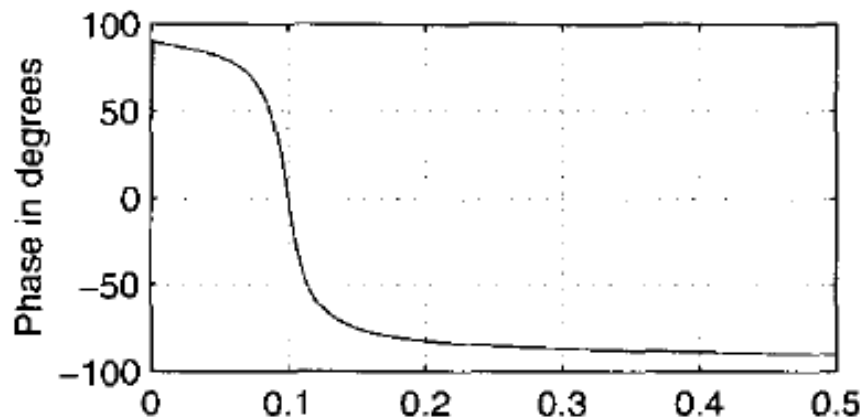
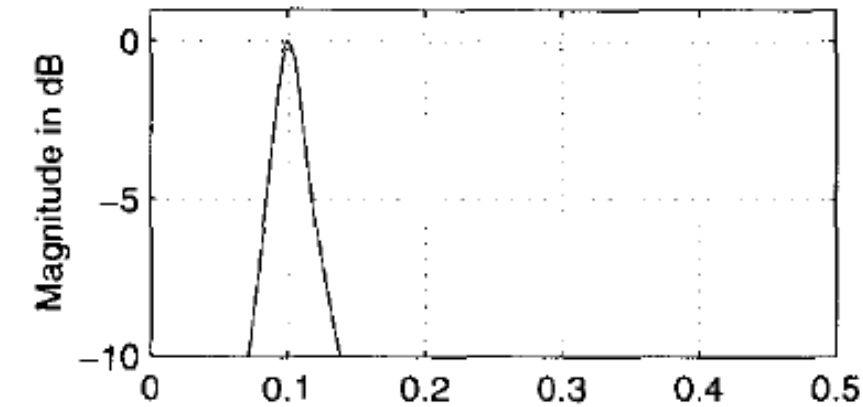
$$A(z) = \frac{-c + d(1 - c)z^{-1} + z^{-2}}{1 + d(1 - c)z^{-1} - cz^{-2}}$$

$$c = \frac{\tan(\pi f_b / f_s) - 1}{\tan(2\pi f_b / f_s) + 1}$$

$$d = -\cos(2\pi f_c / f_s),$$



# Second order band pass/band reject





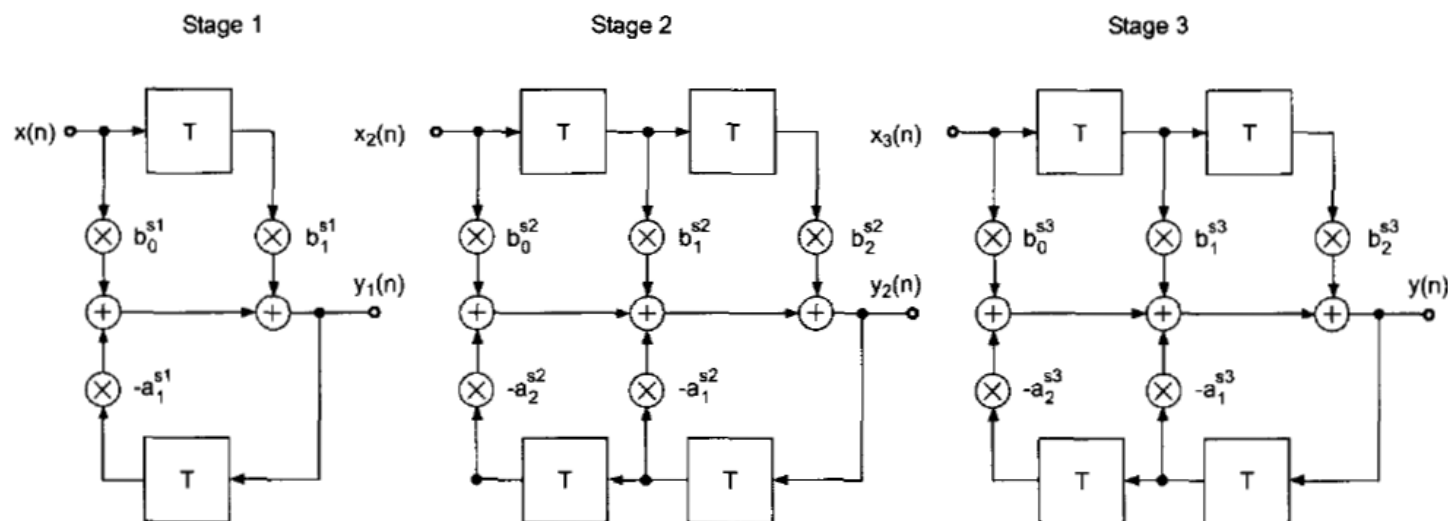
# Series connection of first and second order filters

- If several filters are necessary for spectrum shaping a series connection of first and second order filters is performed

$$H_{1\text{st-order}}(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

$$H_{2\text{nd-order}}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \cdot H_2(z) \cdot H_3(z).$$



# Linear phase filters

- A linear-phase filter is typically used when a causal filter is needed to modify a signal's magnitude-spectrum while preserving the signal's time-domain waveform as much as possible. Linear-phase filters have a symmetric impulse response, e.g.,

$$h(n) = h(N-1-n) \quad n = 0, 1, \dots, N-1$$

- every real symmetric impulse response corresponds to a real frequency response times a linear phase:  $e^{-j\alpha\omega}$  where

$$\alpha = \frac{N-1}{2}$$



# Linear phase filters

- $\alpha$  is the *slope* of the linear phase.
- The filter phase has the form:

$$\Theta(\omega) = -\alpha\omega$$

- Phase delay will be:

$$P(\omega) = -\frac{\Theta(\omega)}{\omega} = -\frac{-\alpha\omega}{\omega} = \alpha$$

- Group delay will be:

$$G(\omega) = -\frac{\partial}{\partial \omega} \Theta(\omega) = \alpha$$