Discrete Time Signals
Discrete Time Fourier Transform

Lesson 2

Discrete Time Signals

A discrete-time signal is represented as a sequence of numbers:

$$x = \{x[n]\}, \qquad -\infty < n < \infty.$$

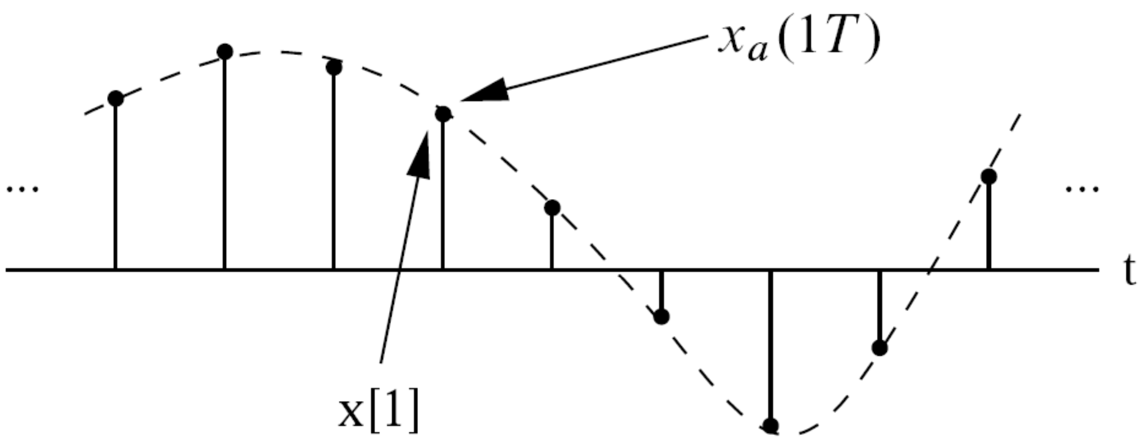
Here n is an integer, and $x[n]$ is the nth sample in the sequence.

Discrete-time signals are often obtained by sampling continuous-time signals. In this case the nth sample of the sequence is equal to the value of the analogue signal $x_a(t)$ at time $t = nT$:

$$x[n] = x_a(nT), \qquad -\infty < n < \infty.$$

Sampling Period

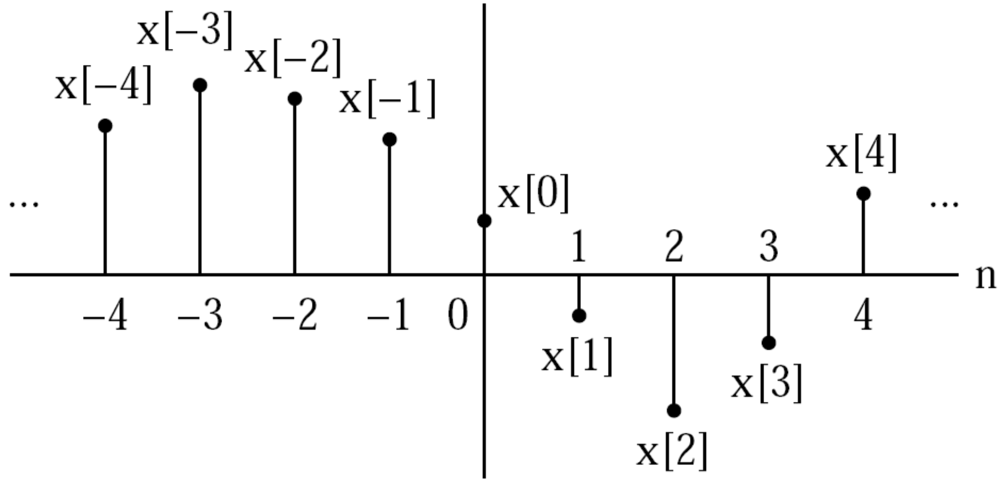
The **sampling period** is then equal to T, and the sampling frequency is $f_s = 1/T$.



For this reason, although $x[n]$ is strictly the nth number in the sequence, we often refer to it as the nth **sample**. We also often refer to "the sequence $x[n]$" when we mean the entire sequence.

Discrete Time Signals

- DTS are often depicted as:



(This can be plotted using the MATLAB function `stem`.) The value $x[n]$ is **undefined** for noninteger values of n.

Sequences can be manipulated in several ways. The **sum** and **product** of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample sum and product respectively. Multiplication of $x[n]$ by a is defined as the multiplication of each sample value by a.

Shifted sequences

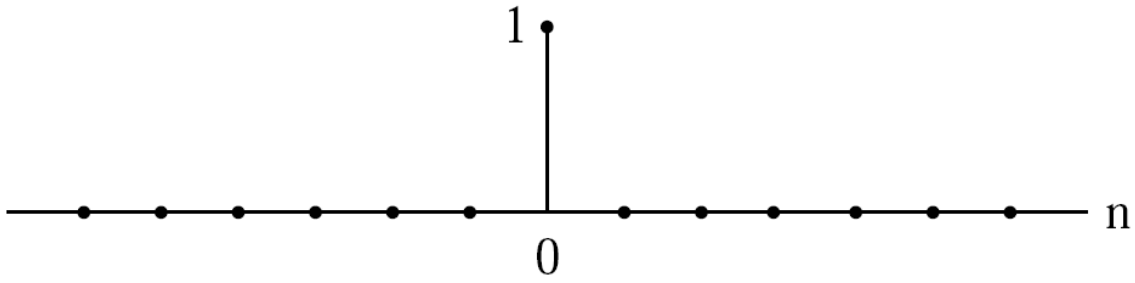
A sequence $y[n]$ is a **delayed** or **shifted** version of $x[n]$ if

$$y[n] = x[n - n_0],$$

with n_0 an integer.

Impulse

The **unit sample sequence**



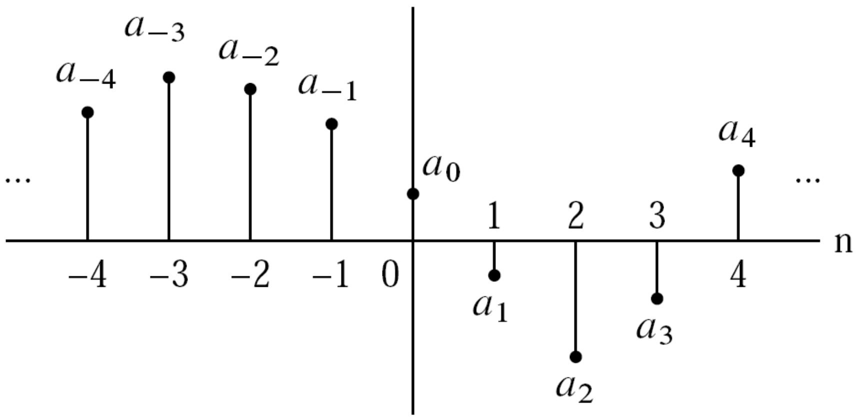
is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0. \end{cases}$$

This sequence is often referred to as a **discrete-time impulse**, or just **impulse**. It plays the same role for discrete-time signals as the Dirac delta function does for continuous-time signals. However, there are no mathematical complications in its definition.

Sequence representation

An important aspect of the impulse sequence is that an arbitrary sequence can be represented as a sum of scaled, delayed impulses. For example, the sequence



can be represented as

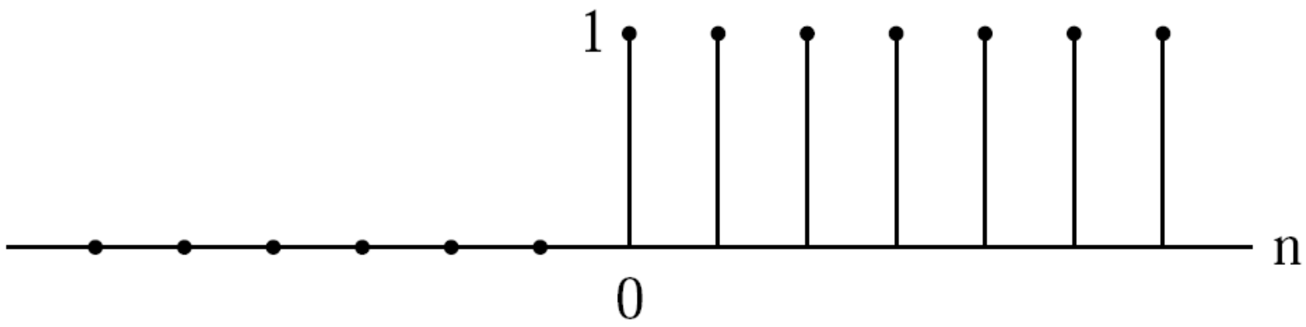
$$x[n] = a_{-4}\delta[n+4] + a_{-3}\delta[n+3] + a_{-2}\delta[n+2] + a_{-1}\delta[n+1] + a_0\delta[n] \\ + a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4].$$

In general, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Unit Step Sequence

The **unit step sequence**



is defined as

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

Unit Step relation with impulse

The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$

Alternatively, this can be expressed as

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots = \sum_{k=0}^{\infty} \delta[n-k].$$

Conversely, the unit sample sequence can be expressed as the first backward difference of the unit step sequence

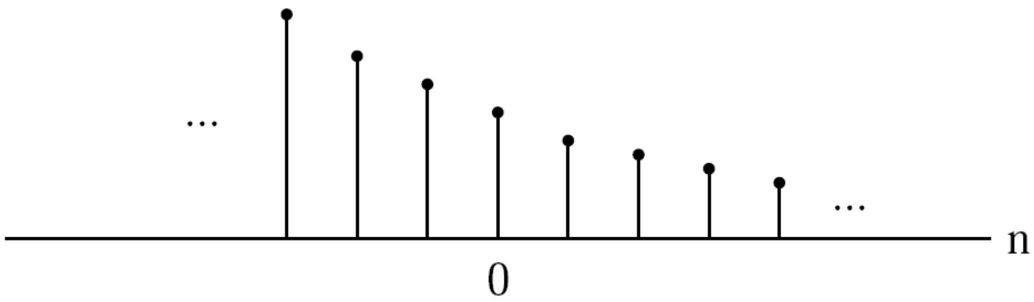
$$\delta[n] = u[n] - u[n-1].$$

Exponential Sequencies

Exponential sequences are important for analysing and representing discrete-time systems. The general form is

$$x[n] = A\alpha^n.$$

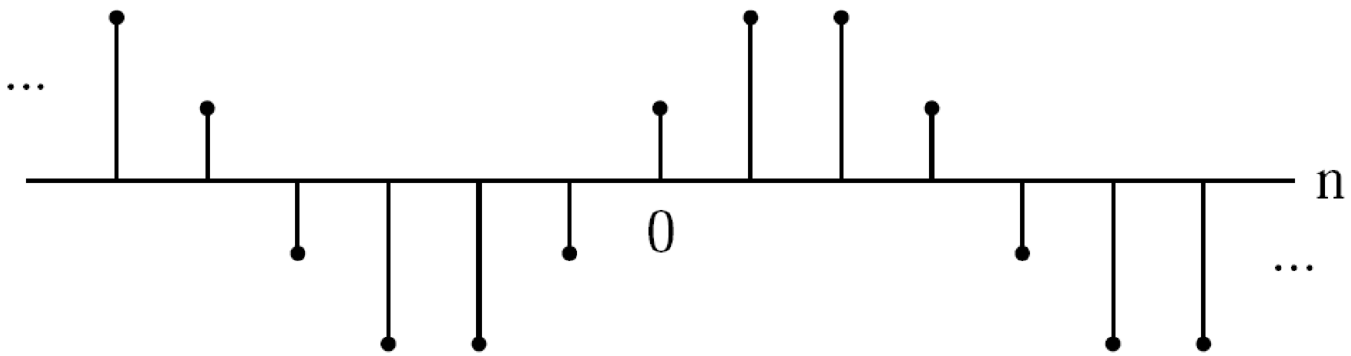
If A and α are real numbers then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n:



For $-1 < \alpha < 0$ the sequence alternates in sign, but decreases in magnitude. For $|\alpha| > 1$ the sequence grows in magnitude as n increases.

Sinusoidal sequence

A **sinusoidal sequence**



Sinusoidal sequences

has the form

$$x[n] = A\cos(\omega_0 n + \phi) \qquad \text{for all } n,$$

with A and ϕ real constants. The exponential sequence $A\alpha^n$ with complex $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$ can be expressed as

$$\begin{aligned} x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} = |A||\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi), \end{aligned}$$

so the real and imaginary parts are exponentially weighted sinusoids.

When $|\alpha| = 1$ the sequence is called the **complex exponential sequence**:

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi).$$

The **frequency** of this complex sinusoid is ω_0, and is measured in radians per sample. The **phase** of the signal is ϕ.

Sinusoidal Sequencies

$$x[n] = A\cos(\omega_0 n + \phi) \qquad \text{for all } n,$$

The index n is always an integer. This leads to some important differences between the properties of discrete-time and continuous-time complex exponentials:

- Consider the complex exponential with frequency $(\omega_0 + 2\pi)$:

$$x[n] = Ae^{j(\omega_0+2\pi)n} = Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}.$$

  Thus the sequence for the complex exponential with frequency ω_0 is *exactly* the same as that for the complex exponential with frequency $(\omega_0 + 2\pi)$. More generally, complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another. Similarly, for sinusoidal sequences

$$x[n] = A\cos[(\omega_0 + 2\pi r)n + \phi] = A\cos(\omega_0 n + \phi).$$

Differences with continuous signals

- In the continuous-time case, sinusoidal and complex exponential sequences are always periodic. Discrete-time sequences are periodic (with period N) if

$$x[n] = x[n+N] \qquad \text{for all } n.$$

Thus the discrete-time sinusoid is only periodic if

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi),$$

which requires that

$$\omega_0 N = 2\pi k \qquad \text{for } k \text{ an integer.}$$

The same condition is required for the complex exponential sequence $Ce^{j\omega_0 n}$ to be periodic.

Continuous vs. discrete sinusoidals

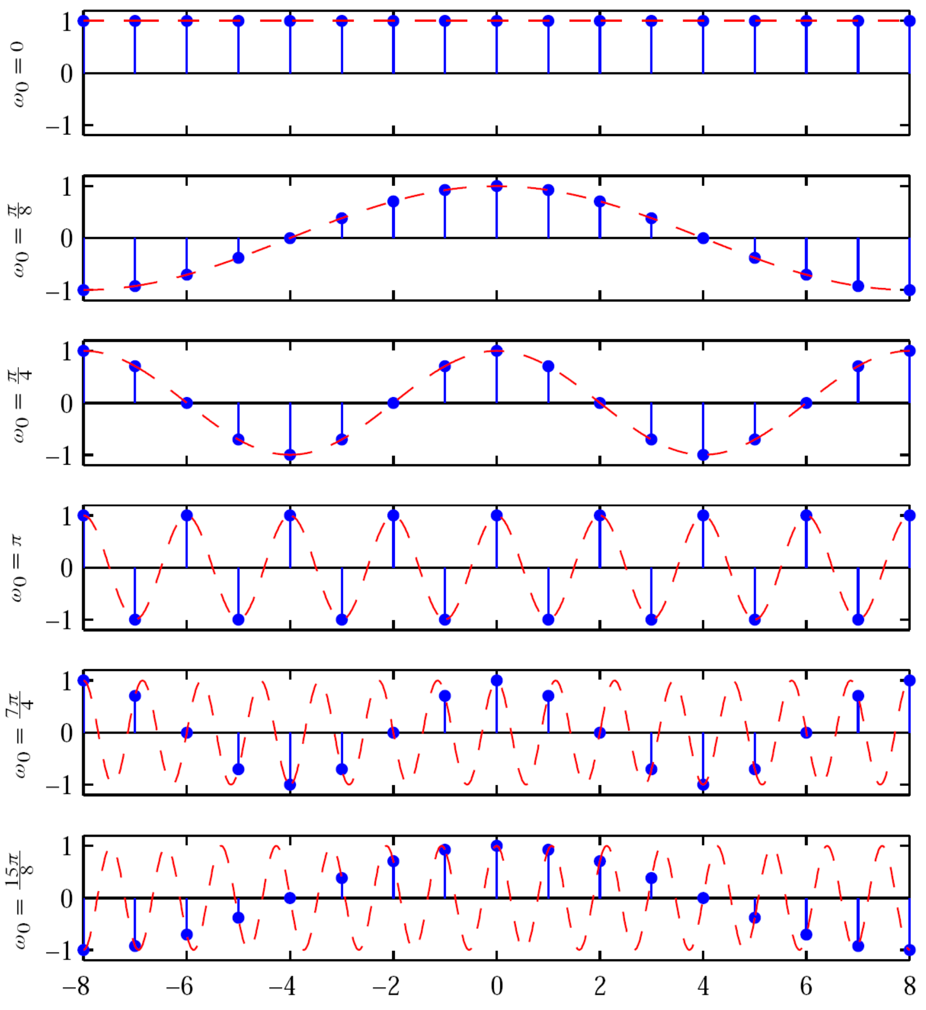
The two factors just described can be combined to reach the conclusion that there are only N distinguishable frequencies for which the corresponding sequences are periodic with period N. One such set is

$$\omega_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \ldots, N-1.$$

Additionally, for discrete-time sequences the interpretation of high and low frequencies has to be modified: the discrete-time sinusoidal sequence $x[n] = A\cos(\omega_0 n + \phi)$ oscillates more rapidly as ω_0 increases from 0 to π, but the oscillations become slower as it increases further from π to 2π.

Continuous vs. discrete sinusoidals

N=16



k=0

k=1

k=2

k=8

k=14

k=15

Discrete Time Systems

A discrete-time system is defined as a transformation or mapping operator that maps an input signal $x[n]$ to an output signal $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\}.$$

Normalized frequencies

- Given the sampling period T the sampling frequency is

$$f_s = \frac{1}{T}$$

- A sampled sinusoid can be written as:

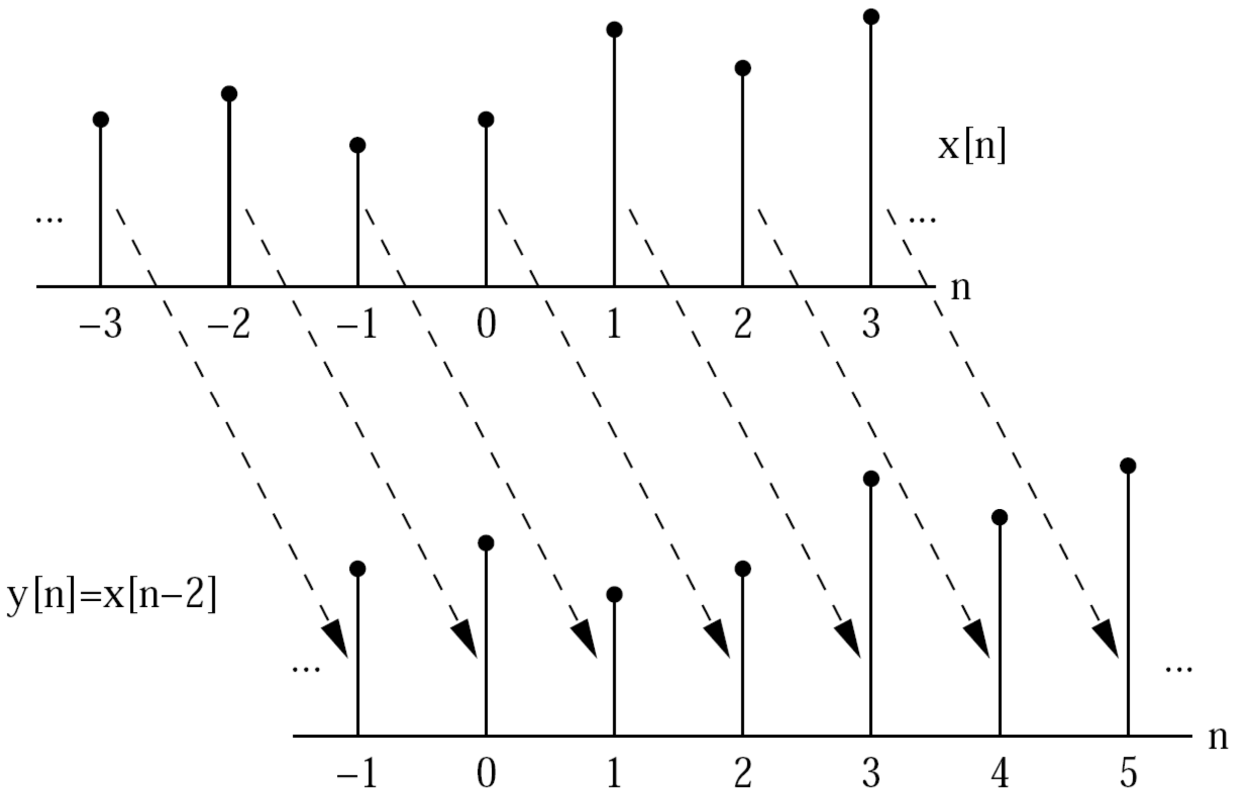
$$x(nT) = A\cos(\omega nT + \phi) = A\cos(2\pi fnT + \phi) =$$

$$= A\cos\left(\frac{2\pi fn}{f_s} + \phi\right) = A\cos\left(\frac{\omega n}{f_s} + \phi\right) = A\cos(\overline{\omega} n + \phi)$$

- $\overline{\omega}$ is the normalized frequency: If the sampling theorem is honoured then $-\pi \le \overline{\omega} \le \pi$

Example: Ideal delay

$$y[n] = x[n - n_d]:$$

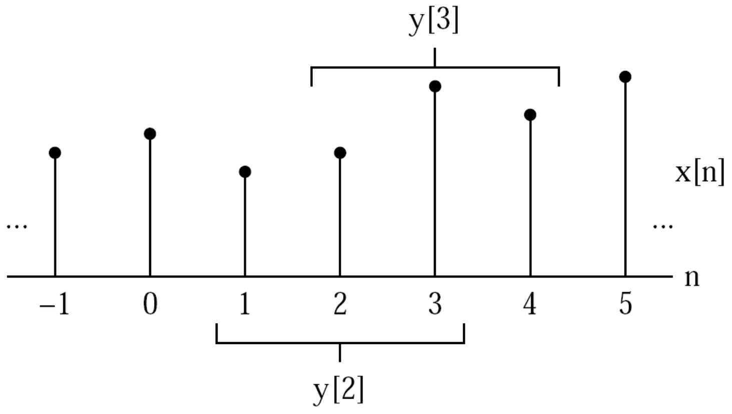


This operation shifts input sequence later by n_d samples.

Example: Moving average

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

For $M_1 = 1$ and $M_2 = 1$, the input sequence



yields an output with

$$\vdots$$

$$y[2] = \frac{1}{3}(x[1] + x[2] + x[3])$$

$$y[3] = \frac{1}{3}(x[2] + x[3] + x[4])$$

$$\vdots$$

In general, systems can be classified by placing constraints on the transformation $T\{\cdot\}$.

Memoryless system

A system is memoryless if the output $y[n]$ depends only on $x[n]$ at the same n. For example, $y[n] = (x[n])^2$ is memoryless, but the ideal delay

$y[n] = x[n - n_d]$ is not unless $n_d = 0$.

Linear Systems

A system is linear if the principle of superposition applies. Thus if $y_1[n]$ is the response of the system to the input $x_1[n]$, and $y_2[n]$ the response to $x_2[n]$, then linearity implies

- **Additivity:**

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$

- **Scaling:**

$$T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n].$$

Linear Systems, Additivity and Scaling

These properties combine to form the general principle of superposition

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} = ay_1[n] + by_2[n].$$

In all cases a and b are arbitrary constants.

This property generalises to many inputs, so the response of a linear system to $x[n] = \sum_k a_k x_k[n]$ will be $y[n] = \sum_k a_k y_k[n]$.

Time invariant Systems

A system is time invariant if a time shift or delay of the input sequence causes a corresponding shift in the output sequence. That is, if $y[n]$ is the response to $x[n]$, then $y[n-n_0]$ is the response to $x[n-n_0]$.

is time invariant, but the compressor system

$$y[n] = x[Mn]$$

for M a positive integer (which selects every Mth sample from a sequence) is not.

Causality

A system is causal if the output at n depends only on the input *at* n *and earlier inputs.*

For example, the backward difference system

$$y[n] = x[n] - x[n-1]$$

is causal, but the forward difference system

$$y[n] = x[n+1] - x[n]$$

is not.

Stability

A system is stable if every bounded input sequence produces a bounded output sequence:

- **Bounded input:** $|x[n]| \leq B_x < \infty$
- **Bounded output:** $|y[n]| \leq B_y < \infty$.

For example, the accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[n]$$

is an example of an *unbounded* system, since its response to the unit step $u[n]$ is

$$y[n] = \sum_{k=-\infty}^{n} u[n] = \begin{cases} 0 & n < 0 \\ n+1 & n \geq 0, \end{cases}$$

which has no finite upper bound.

Linear Time Invariant Systems

If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses, then it follows that a linear time-invariant (LTI) system can be completely characterised by its impulse response.

Suppose $h_k[n]$ is the response of a linear system to the impulse $\delta[n-k]$ at $n = k$. Since

$$y[n] = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\},$$

the principle of superposition means that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n].$$

The convolution operation for LTI

If the system is additionally time invariant, then the response to $\delta[n-k]$ is $h[n-k]$. The previous equation then becomes

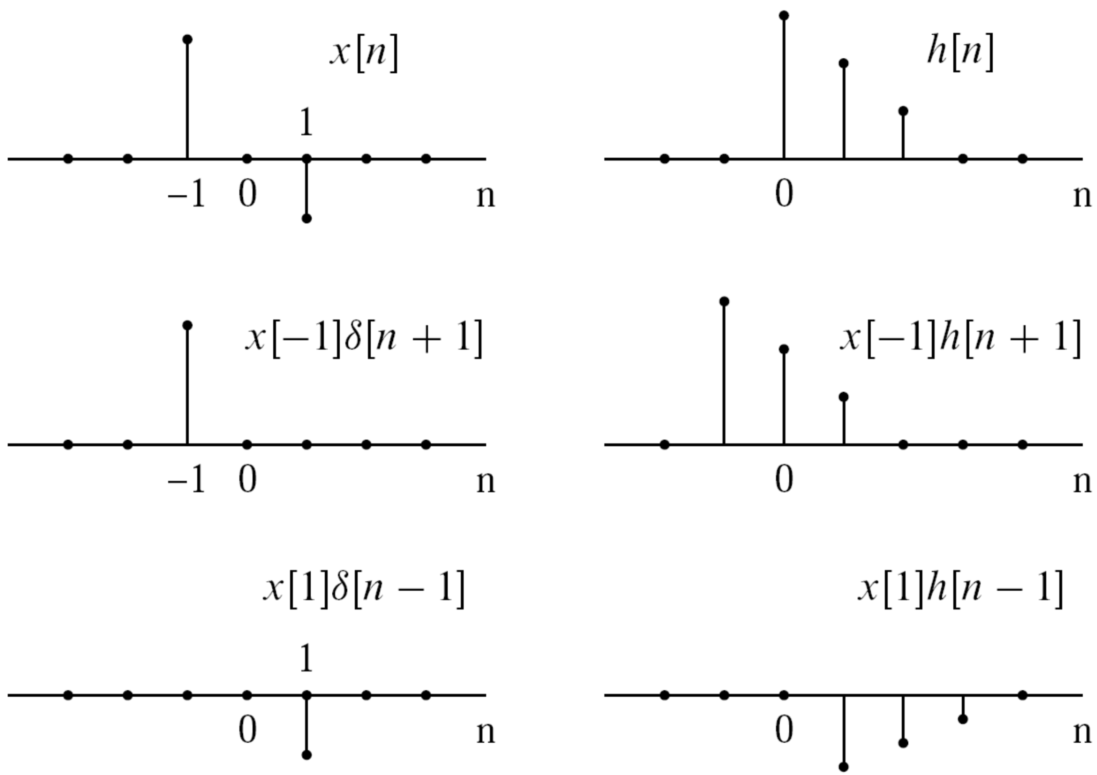
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

This expression is called the **convolution sum**. Therefore, a LTI system has the property that given $h[n]$, we can find $y[n]$ for *any* input $x[n]$. Alternatively, $y[n]$ is the **convolution** of $x[n]$ with $h[n]$, denoted as follows:

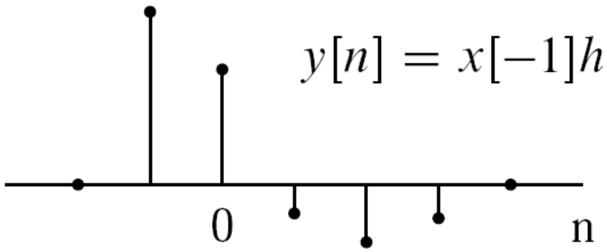
$$y[n] = x[n] * h[n].$$

Convolution

The previous derivation suggests the interpretation that the input sample at $n = k$, represented by $x[k]\delta[n-k]$, is transformed by the system into an output sequence $x[k]h[n-k]$. For each k, these sequences are superimposed to yield the overall output sequence:



Convolution

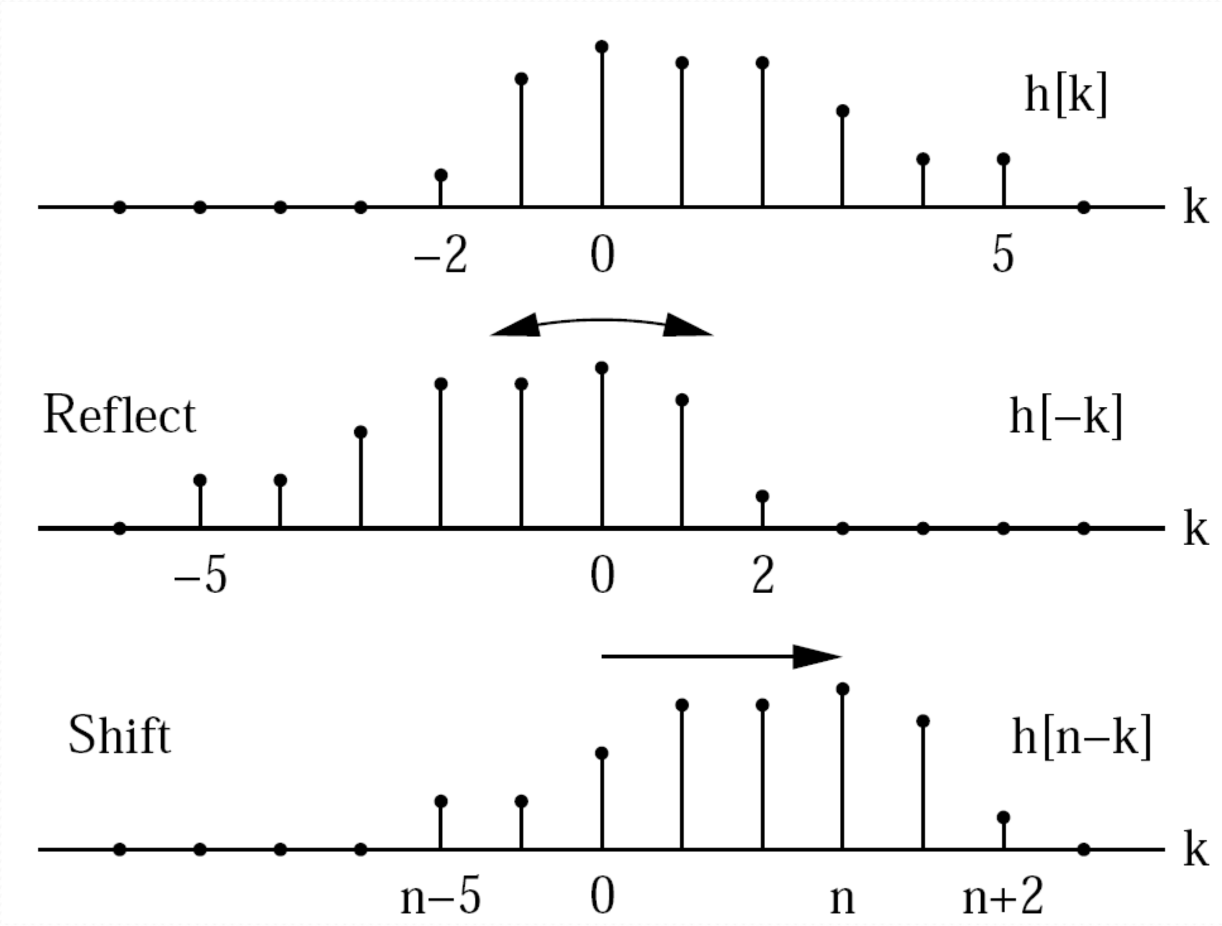


$$y[n] = x[-1]h[n+1] + x[1]h[n-1]$$

A slightly different interpretation, however, leads to a convenient computational form: the nth value of the output, namely $y[n]$, is obtained by multiplying the input sequence (expressed as a function of k) by the sequence with values $h[n-k]$, and then summing all the values of the products $x[k]h[n-k]$. The key to this method is in understanding how to form the sequence $h[n-k]$ for all values of n of interest.

To this end, note that $h[n-k] = h[-(k-n)]$. The sequence $h[-k]$ is seen to be equivalent to the sequence $h[k]$ reflected around the origin:

Convolution



Convolution

The sequence $h[n-k]$ is then obtained by shifting the origin of the sequence to $k=n$.

To implement discrete-time convolution, the sequences $x[k]$ and $h[n-k]$ are multiplied together for $-\infty < k < \infty$, and the products summed to obtain the value of the output sample $y[n]$. To obtain another output sample, the procedure is repeated with the origin shifted to the new sample position.

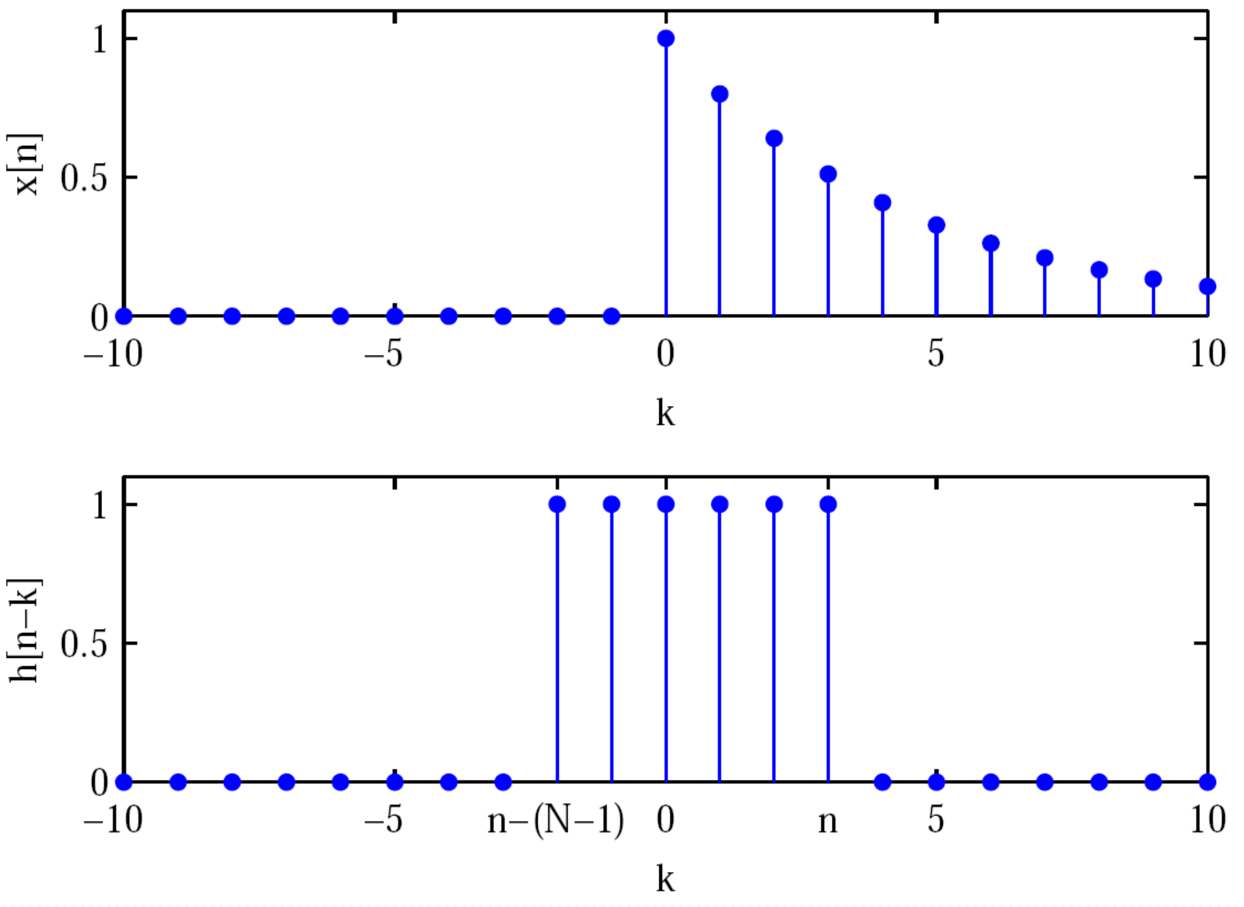
Convolution example

Consider the output of a system with impulse response

$$h[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

to the input $x[n] = a^n u[n]$. To find the output at n, we must form the sum over all k of the product $x[k]h[n-k]$.

Convolution example



Convolution example

Since the sequences are non-overlapping for all negative n, the output must be zero:

$$y[n] = 0, \qquad n < 0.$$

For $0 \leq n \leq N - 1$ the product terms in the sum are $x[k]h[n-k] = a^k$, so it follows that

$$y[n] = \sum_{k=0}^{n} a^k, \qquad 0 \leq n \leq N - 1.$$

Finally, for $n > N - 1$ the product terms are $x[k]h[n-k] = a^k$ as before, but the lower limit on the sum is now $n - N + 1$. Therefore

$$y[n] = \sum_{k=n-N+1}^{n} a^k, \qquad n > N - 1.$$

Properties of LTI Systems

All LTI systems are described by the convolution sum

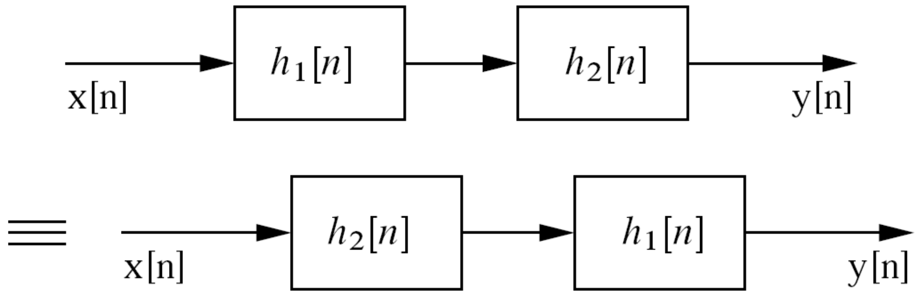
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

Some properties of LTI systems can therefore be found by considering th properties of the convolution operation:

- **Commutative:** $x[n] * h[n] = h[n] * x[n]$
- **Distributive over addition:**

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

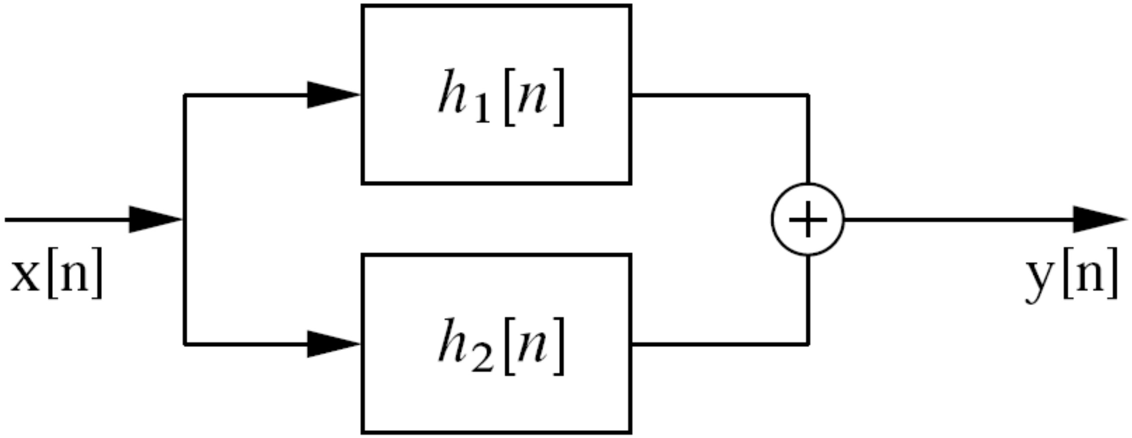
- **Cascade connection:**



$$y[n] = h[n] * x[n] = h_1[n] * h_2[n] * x[n] = h_2[n] * h_1[n] * x[n].$$

Properties of LTI Systems

- **Parallel connection:**



$$y[n] = (h_1[n] + h_2[n]) * x[n] = h_p[n] * x[n].$$

Additional important properties are:

- A LTI system is **stable** if and only if $S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

Properties of LTI Systems

The **ideal delay** system $h[n] = \delta[n - n_d]$ is stable since $S = 1 < \infty$; the **moving average** system

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$

$$= \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise,} \end{cases}$$

the **forward difference** system $h[n] = \delta[n+1] - \delta[n]$, and the **backward difference** system $h[n] = \delta[n] - \delta[n-1]$ are stable since S is the sum of a finite number of finite samples, and is therefore less than ∞; the **accumulator** system

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$= \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$= u[n]$$

is unstable since $S = \sum_{n=0}^{\infty} u[n] = \infty$.

FIR and IIR

Systems with only a finite number of nonzero values in $h[n]$ are called **Finite duration impulse response (FIR)** systems. FIR systems are stable if each impulse response value is finite. The ideal delay, the moving average, and the forward and backward difference described above fall into this class. **Infinite impulse response (IIR)** systems, such as the accumulator system, are more difficult to analyse. For example, the accumulator system is unstable, but the IIR system

$$h[n] = a^n u[n], \qquad |a| < 1$$

is stable since

$$S = \sum_{n=0}^{\infty} |a^n| \leq \sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty$$

(it is the sum of an infinite geometric series).

Geometric series

- The general formula for converging geometric series is:

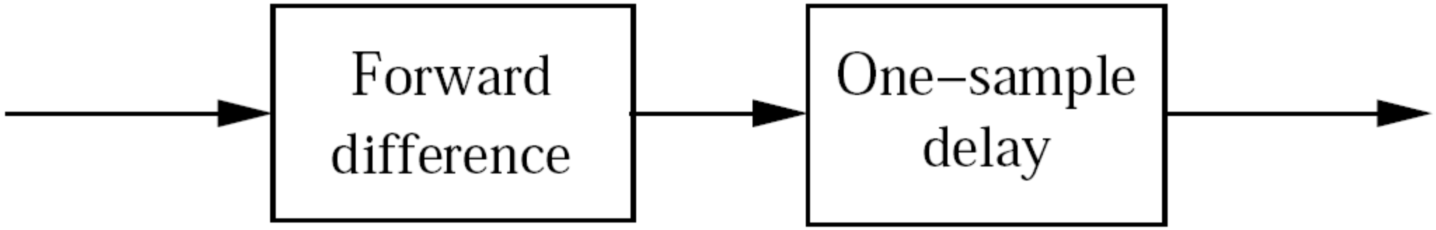
$$\sum_{n=0}^{k} q^n = \frac{1-q^{k+1}}{1-q}$$

- For infinite series the convergence request is: $|q|<1$

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q}.$$

Causality

Consider the system

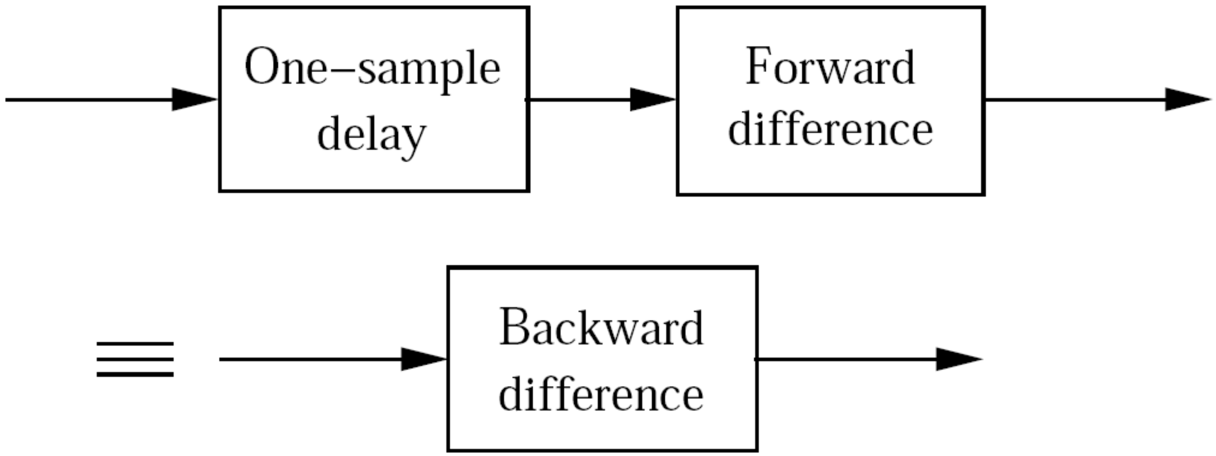


which has

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n-1] * \delta[n+1] - \delta[n-1] * \delta[n] \\ &= \delta[n] - \delta[n-1]. \end{aligned}$$

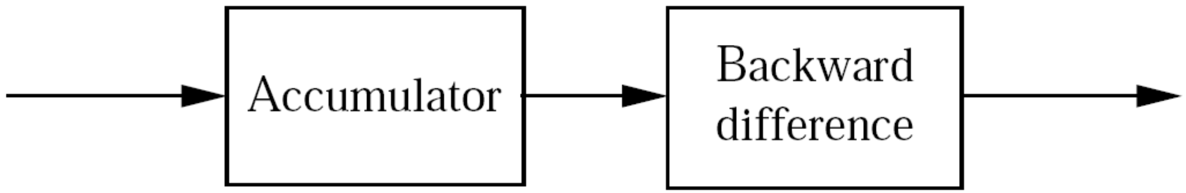
Causality by a delay

This is the impulse response of a backward difference system:



Here a non-causal system has been converted to a causal one by cascading with a delay. In general, *any non-causal FIR system can be made causal by cascading with a sufficiently long delay.*

Inverse system example



The impulse response of this system is

$$h[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n].$$

The output is therefore equal to the input because $x[n] * \delta[n] = x[n]$. Thus the backward difference exactly compensates for (or inverts) the effect of the accumulator — the backward difference system is the **inverse system** for the accumulator, and vice versa. We define this inverse relationship for all LTI systems:

$$h[n] * h_i[n] = \delta[n].$$

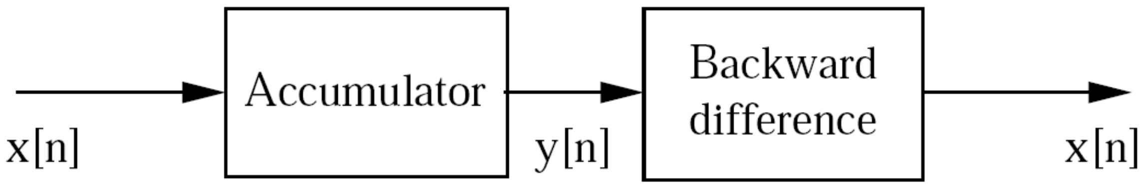
Linear Constant Coefficient Difference Equations

Some LTI systems can be represented in terms of linear constant coefficient difference (LCCD) equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m].$$

Example: difference equation representation of the accumulator

Take for example the accumulator

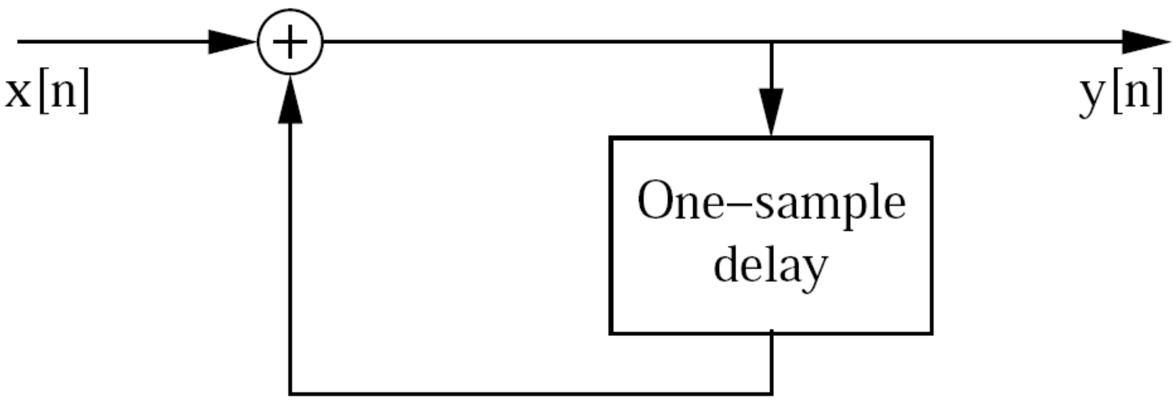


Here $y[n] - y[n-1] = x[n]$, which can be written in the desired form with $N = 1$, $a_0 = 1$, $a_1 = -1$, $M = 0$, and $b_0 = 1$. Rewriting as

$$y[n] = y[n-1] + x[n]$$

Recursive Representation

we obtain the **recursion representation**



where at n we add the current input $x[n]$ to the previously accumulated sum $y[n-1]$.

Difference equation for moving average

Consider now the moving average system with $M_1 = 0$:

$$h[n] = \frac{1}{M_2 + 1}(u[n] - u[n - M_2 - 1]).$$

The output of the system is

$$y[n] = \frac{1}{M_2 + 1}\sum_{k=0}^{M_2} x[n - k],$$

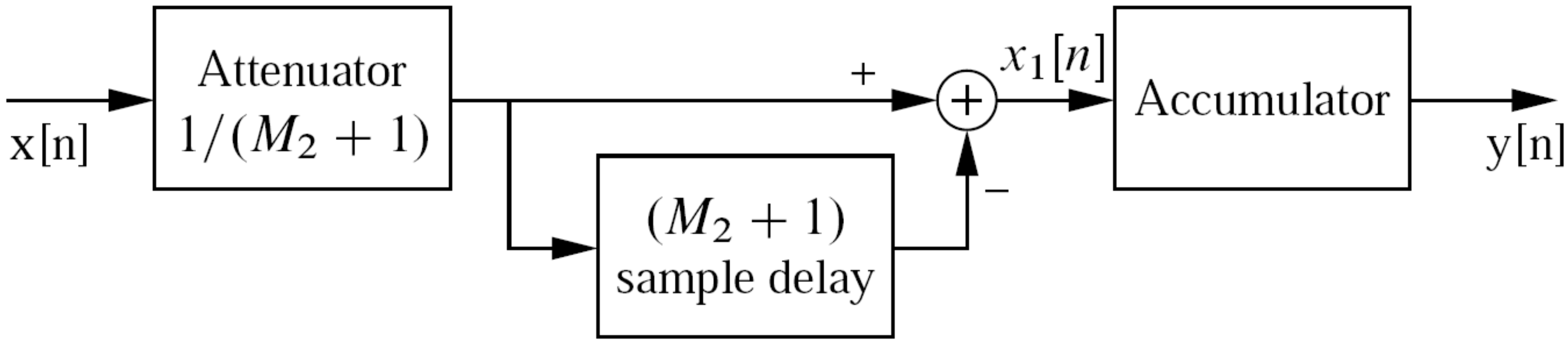
which is a LCCDE with $N = 0$, $a_0 = 1$, and $M = M_2$, $b_k = 1/(M_2 + 1)$.

Difference equation for moving average

Using the sifting property of $\delta[n]$,

$$h[n] = \frac{1}{M_2 + 1}(\delta[n] - \delta[n - M_2 - 1]) * u[n]$$

so



Difference Equations

Here $x_1[n] = 1/(M_2 + 1)(x[n] - x[n - M_2 - 1])$ and for the accumulator $y[n] - y[n-1] = x_1[n]$. Therefore

$$y[n] - y[n-1] = \frac{1}{M_2 + 1}(x[n] - x[n - M_2 - 1]),$$

which is again a (different) LCCD equation with $N = 1$, $a_0 = 1$, $a_1 = -1$, $b_0 = -b_{M_2+1} = 1/(M_2 + 1)$.

Difference equations

As for constant coefficient differential equations in the continuous case, without additional information or constraints a LCCDE does not provide a unique solution for the output given an input. Specifically, suppose we have the particular output $y_p[n]$ for the input $x_p[n]$. The same equation then has the solution

$$y[n] = y_p[n] + y_h[n],$$

where $y_h[n]$ is any solution with $x[n] = 0$. That is, $y_h[n]$ is an **homogeneous solution** to the **homogeneous equation**

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0.$$

It can be shown that there are N nonzero solutions to this equation, so a set of N auxiliary conditions are required for a unique specification of $y[n]$ for a given $x[n]$.

If a system is LTI *and causal*, then the initial conditions are **initial rest** conditions, and a unique solution can be obtained.

Frequency domain Representation for DTS

The Fourier transform considered here is strictly speaking the **discrete-time Fourier transform (DTFT)**, although Oppenheim and Schafer call it just the Fourier transform. Its properties are recapped here (with examples) to show nomenclature.

The Fourier Transform of the impulse response is called the frequency response:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}.$$

Frequency response for delay

Consider the input $x[n] = e^{j\omega n}$ to the ideal delay system $y[n] = x[n - n_d]$: the output is

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}.$$

The frequency response is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}.$$

Alternatively, for the ideal delay $h[n] = \delta[n - n_d]$,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}.$$

The real and imaginary parts of the frequency response are

Delay: amplitude and phase

$H_R(e^{j\omega}) = \cos(\omega n_d)$ and $H_I(e^{j\omega}) = \sin(\omega n_d)$, or alternatively

$$|H(e^{j\omega})| = 1$$

$$\sphericalangle H(e^{j\omega}) = -\omega n_d .$$

The frequency response of a LTI system is essentially the same for continuous and discrete time systems. However, an important distinction is that in the discrete case it is *always* periodic in frequency with a period 2π:

$$\begin{aligned} H(e^{j(\omega+2\pi)}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n} \end{aligned}$$

Delay: amplitude and phase

$$= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = H(e^{j\omega}).$$

This last result holds since $e^{\pm j2\pi n} = 1$ for integer n.

The reason for this periodicity is related to the observation that the sequence

$$\left\{e^{-j\omega n}\right\}, \qquad -\infty < n < \infty$$

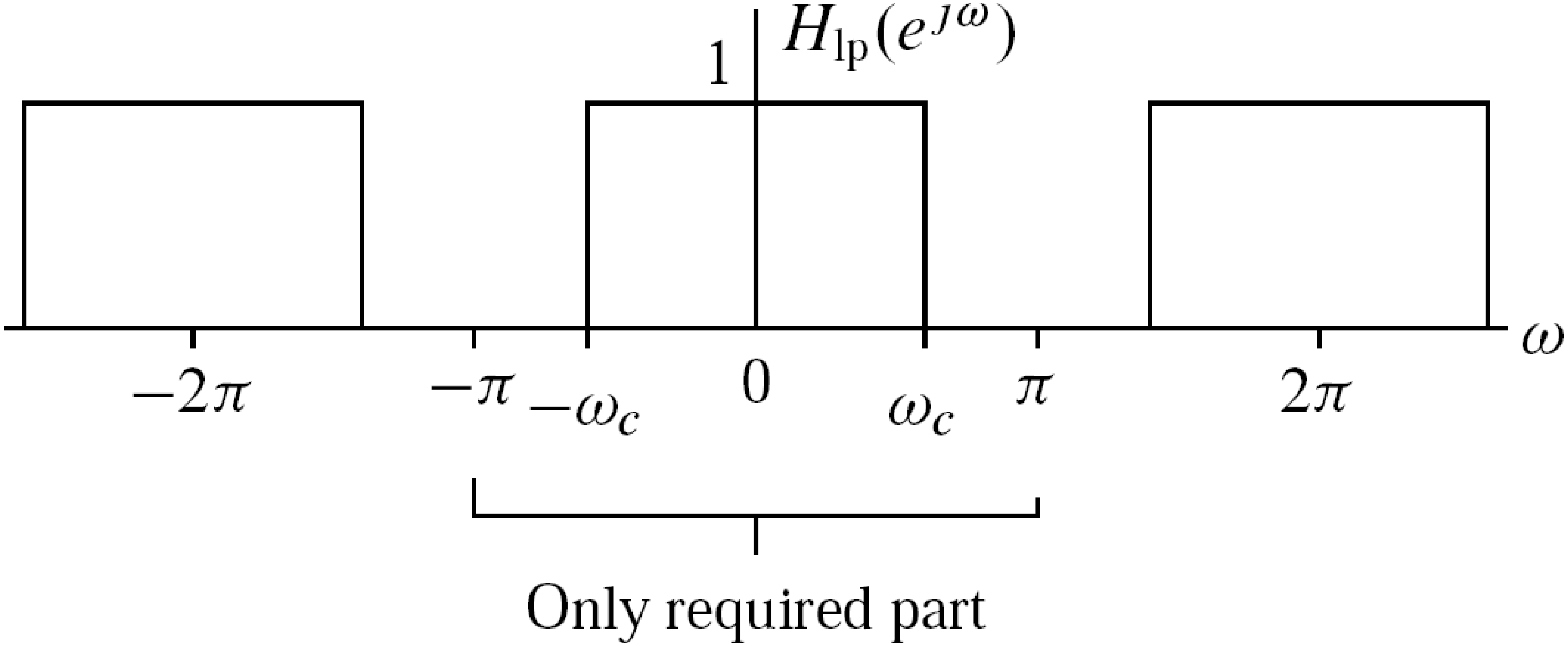
has exactly the same values as the sequence

$$\left\{e^{-j(\omega+2\pi)n}\right\}, \qquad -\infty < n < \infty.$$

A system will therefore respond in exactly the same way to both sequences.

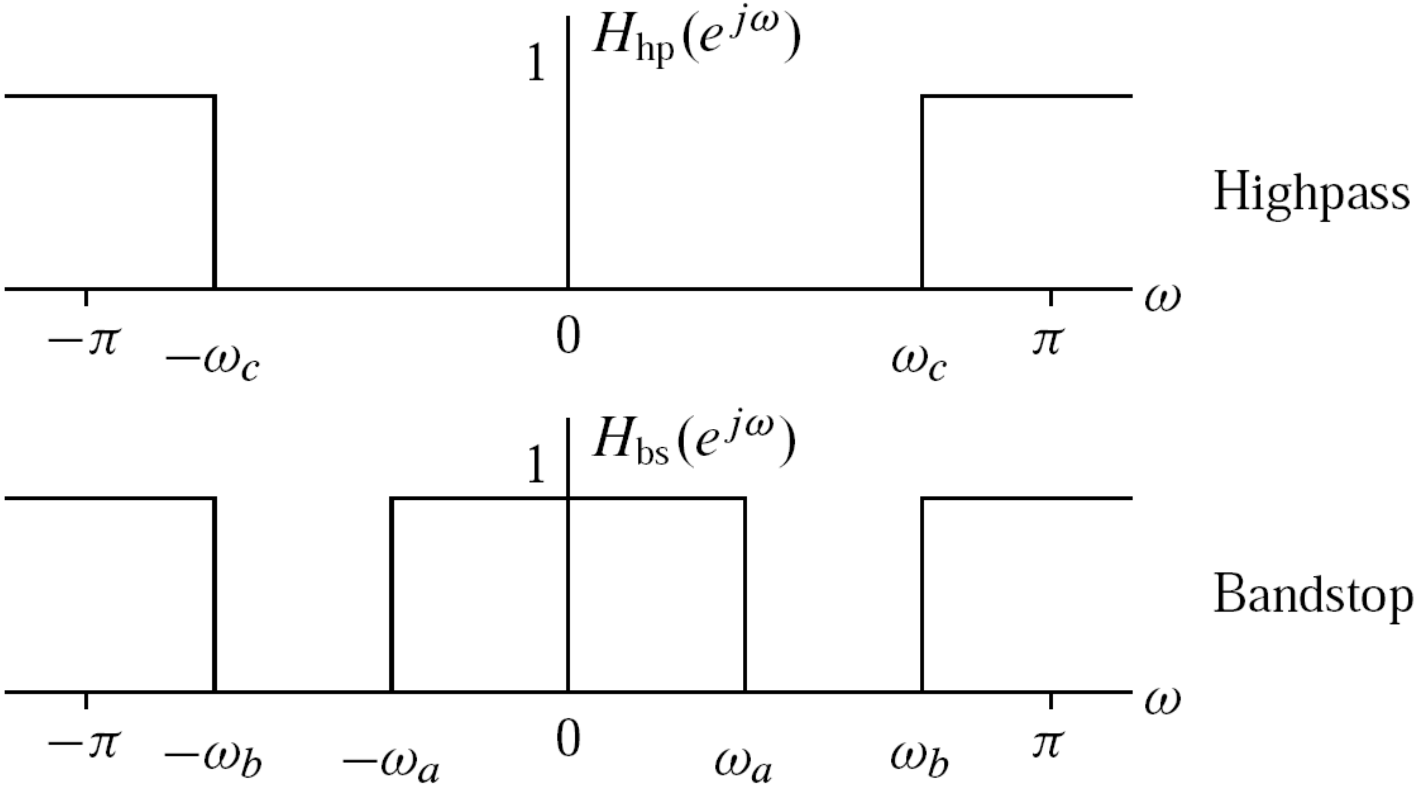
Ideal filters for discrete signals

The frequency response of an ideal lowpass filter is as follows:

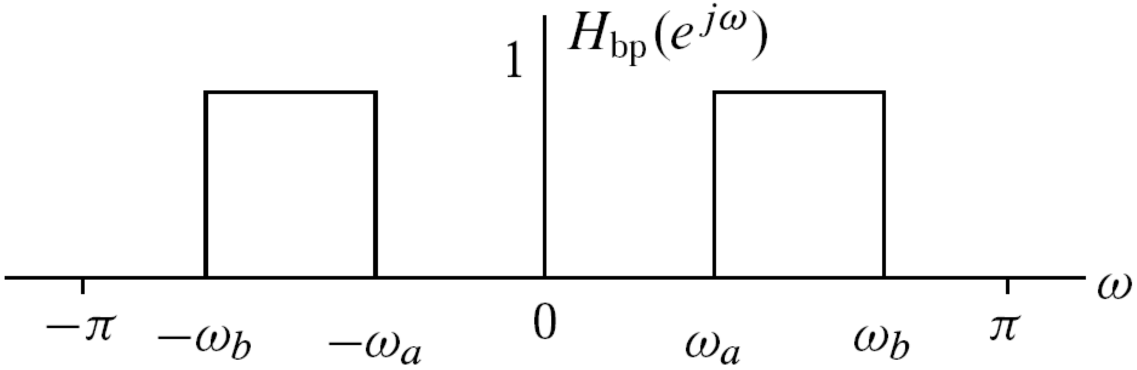


Ideal filters for discrete signals

Due to the periodicity in the response, it is only necessary to consider on[e] frequency cycle, usually chosen to be the range $-\pi$ to π. Other example[s] ideal filters are:



Ideal filters for discrete signals



Bandpass

In these cases it is implied that the frequency response repeats with period 2π outside of the plotted interval.

Frequency response of a moving average system

The frequency response of the moving average system

$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1} & -M_1 \leq n \leq M_2 \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_2+M_1+1)/2} - e^{-j\omega(M_2+M_1+1)/2}}{1 - e^{-j\omega}} e^{-\frac{j\omega(M_2-M_1+1)}{2}} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega(M_2+M_1+1)/2} - e^{-j\omega(M_2+M_1+1)/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-\frac{j\omega(M_2-M_1)}{2}} \\ &= \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-\frac{j\omega(M_2-M_1)}{2}}. \end{aligned}$$

Continued

- Since

$$1 - e^{j\alpha} = e^{j\alpha/2}\left(e^{-j\alpha/2} - e^{j\alpha/2}\right) = e^{j\alpha/2}\,2j\,\frac{e^{-j\alpha/2} - e^{j\alpha/2}}{2j} =$$

$$= -2\sin\left(\frac{\alpha}{2}\right)e^{j\alpha/2}j = -2\sin\left(\frac{\alpha}{2}\right)e^{j\alpha/2}e^{j\pi/2} =$$

$$= 2\sin\left(\frac{\alpha}{2}\right)e^{j\alpha/2}e^{j\pi/2}e^{-j\pi} = 2\sin\left(\frac{\alpha}{2}\right)e^{j\frac{\alpha-\pi}{2}}$$

Continued

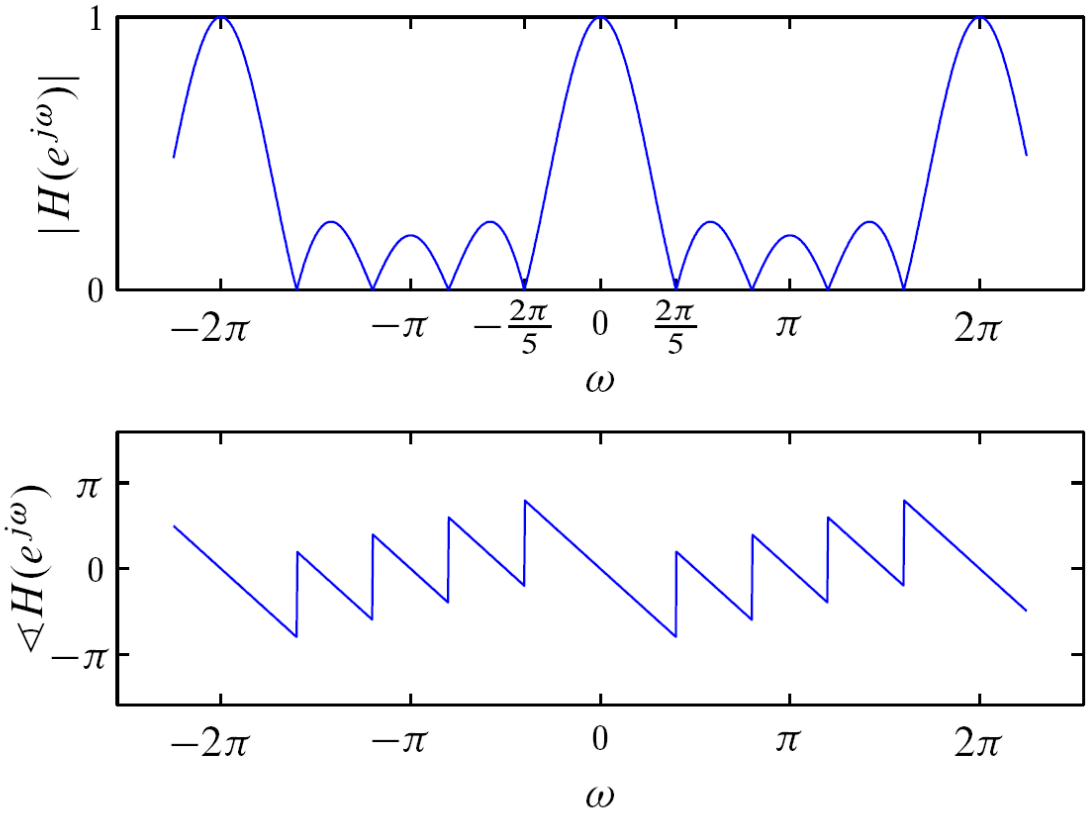
$$\sum_{n=-M_1}^{M_2} e^{-j\omega n} = \left(\sum_{n=0}^{M_1+M_2} e^{-j\omega n} \right) e^{j\omega M_1} = \frac{1-e^{-j\omega(M_1+M_2+1)}}{1-e^{-j\omega}} e^{j\omega M_1} =$$

$$= \frac{e^{-j\omega\frac{(M_1+M_2+1)}{2}}}{e^{-j\frac{\omega}{2}}} \frac{e^{j\omega\frac{(M_1+M_2+1)}{2}} - e^{-j\omega\frac{(M_1+M_2+1)}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} e^{j\omega M_1} =$$

$$= \frac{\sin\left(\omega \frac{(M_1+M_2+1)}{2} \right)}{\sin\frac{\omega}{2}} e^{-j\omega\frac{M_1+M_2+1-2M_1-1}{2}}$$

Frequency response of a moving average system

For $M_1 = 0$ and $M_2 = 4$,



Discrete Time Fourier Transform (DTFT)

The discrete time Fourier transform (DTFT) of the sequence $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

This is also called the **forward transform** or **analysis** equation. The **inverse Fourier transform**, or **synthesis** formula, is given by the Fourier integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega.$$

The Fourier transform is generally a complex-valued function of ω:

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}.$$

Discrete Time Fourier Transform (DTFT)

The quantities $|X(e^{j\omega})|$ and $\sphericalangle X(e^{j\omega})$ are referred to as the **magnitude** and **phase** of the Fourier transform. The Fourier transform is often referred to as the **Fourier spectrum**.

Since the frequency response of a LTI system is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k},$$

it is clear that the frequency response is equivalent to the Fourier transform of the impulse response, and the impulse response is

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega.$$

Discrete Time Fourier Transform (DTFT)

A sufficient condition for the existence of the Fourier transform of a sequence $x[n]$ is that it be absolutely summable: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. In other words, the Fourier transform exists if the sum $\sum_{n=-\infty}^{\infty} |x[n]|$ converges. The Fourier transform may however exist for sequences where this is not true — a rigorous mathematical treatment can be found in the theory of **generalised functions**.

Symmetry properties

Any sequence $x[n]$ can be expressed as

$$x[n] = x_e[n] + x_o[n],$$

where $x_e[n]$ is **conjugate symmetric** $(x_e[n] = x_e^*[-n])$ and $x_o[n]$ is **conjugate antisymmetric** $(x_o[n] = -x_o^*[-n])$. These two components of the sequence can be obtained as:

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$
$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n].$$

If a real sequence is conjugate symmetric, then it is an **even** sequence, and if conjugate antisymmetric, then it is **odd**.

example

- x=[1, 2, -3, 1]

  0

$$x_e = \left[1/2, -1, 2, -1, 1/2\right]$$

$$x_o = \left[-1/2, 2, 0, -2, 1/2\right]$$

Symmetry properties

Similarly, the Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate symmetric and antisymmetric parts:

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}),$$

where

$$X_e(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$

$$X_o(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})].$$

With these definitions, and letting

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}),$$

Symmetry properties

Sequence $x[n]$	Transform $X(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_e(e^{j\omega})$
$j\,\mathrm{Im}\{x[n]\}$	$X_o(e^{j\omega})$
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

Symmetry properties

Most of these properties can be proved by substituting into the expression for the Fourier transform. Additionally, for real $x[n]$ the following also hold:

Real sequence $x[n]$	Transform $X(e^{j\omega})$
$x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
$x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$
$x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$
$x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$
$x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$
$x_e[n]$	$X_R(e^{j\omega})$
$x_o[n]$	$jX_I(e^{j\omega})$

Properties of the Fourier transform

Sequences $x[n]$, $y[n]$	Transforms $X(e^{j\omega})$, $Y(e^{j\omega})$	Property
$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$	Linearity
$x[n - n_d]$	$e^{-j\omega n_d} X(e^{j\omega})$	Time shift
$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$	Frequency shift
$x[-n]$	$X(e^{-j\omega})$	Time reversal
$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$	Frequency diff.
$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	Convolution
$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	Modulation

Useful transform pairs

Sequence	Fourier transform
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-j\omega n_0}$
$1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \lvert\omega\rvert < \omega_c \\ 0 & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$