Classical Signal Theory

Lesson 1

Continuous Signals

- A continuous-time signal is a complex function of a real variable that has, as a codomain, the set of complex numbers.

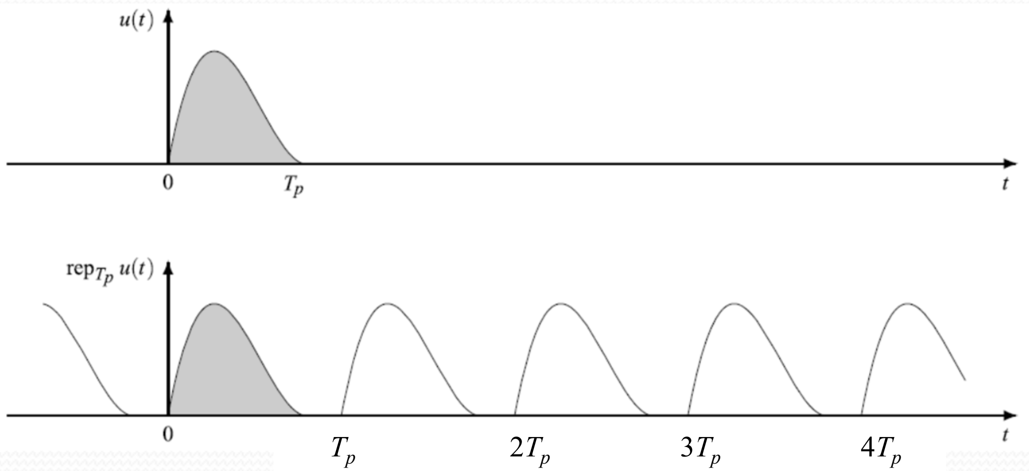
$$s(t),\ t \in \mathbb{R}$$

- Real signals: $s(t) = s^*(t)$

Periodic Signals

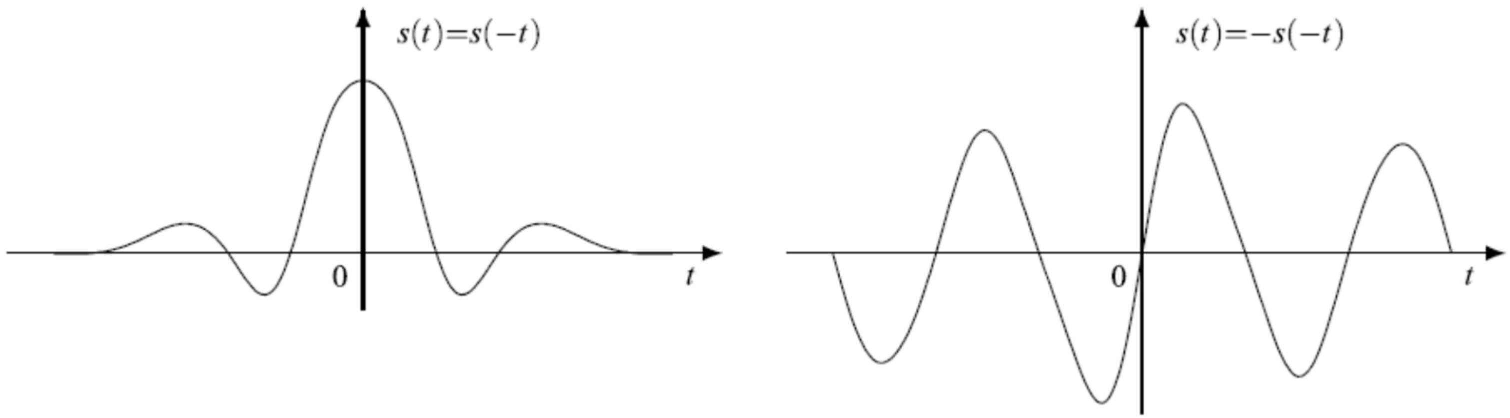
- Periodic signals: $s(t+T_p)=s(t),$
  where the condition is satisfied for T_p and for kT_p where k is an integer.
- Periodic repetition formulation:

$$s(t)=\sum_{n=-\infty}^{+\infty} u(t-nT_p) \overset{\Delta}{=} \mathrm{rep}_{T_p}\, u(t),$$



Continuous Signals

- A signal is *even* if: $s(-t) = s(t),$
- A signal is *odd* if: $s(-t) = -s(t)$

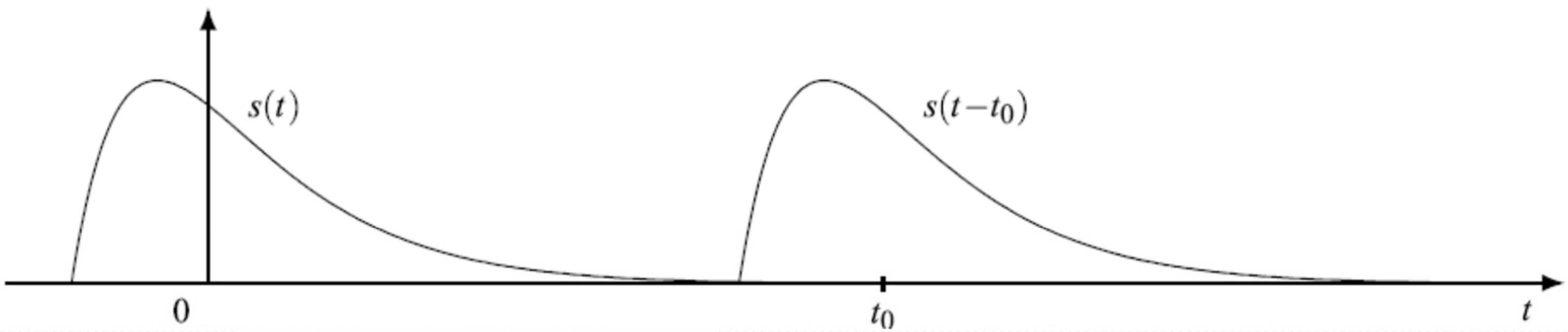


- An arbitrary signal can be always decomposed into the sum of an *even* component $s_e(t)$ and an *odd* component $s_o(t)$
- $$s(t) = s_e(t) + s_o(t),$$

$$s_e(t) = \frac{1}{2}[s(t) + s(-t)], \qquad s_o(t) = \frac{1}{2}[s(t) - s(-t)].$$

Continuous Signals

- Causal signal: $s(t) = 0 \quad \text{for } t < 0.$
- Time shift: $s_{t_0}(t) = s(t - t_0)$



- Area: $\text{area}(s) = \int_{-\infty}^{+\infty} s(t)\,\mathrm{d}t.$
- Mean value: $m_s = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} s(t)\,\mathrm{d}t$

Continuous Signals

- Energy: $$E_s = \int_{-\infty}^{+\infty} \left| s(t) \right|^2 \mathrm{d}t,$$
- Specific power: $$P_s = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| s(t) \right|^2 \mathrm{d}t.$$
-

Definitions over a period

- Mean value over a period: $$m_s(T_p) = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} s(t)\,\mathrm{d}t.$$
- Energy over a period: $$E_s(T_p) = \int_{t_0}^{t_0+T_p} |s(t)|^2\,\mathrm{d}t.$$
- Power over a period: $$P_s(T_p) = \frac{1}{T_p}E_s(T_p) = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} |s(t)|^2\,\mathrm{d}t.$$

Example of a signal

- A sinusoidal signal:

$$s(t) = A_0 \cos(\omega_0 t + \phi_0) = A_0 \cos(2\pi f_0 t + \phi_0) = A_0 \cos\left(2\pi \frac{t}{T_0} + \phi_0\right)$$

  - It can be written as: $s(t) = A_0 \cos\phi_0 \cos\omega_0 t - A_0 \sin\phi_0 \sin\omega_0 t,$
  - Using Euler's formulas:

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}, \quad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

  - It becomes:

$$s(t) = A_0 \cos(\omega_0 t + \phi_0) = \frac{1}{2} A_0 e^{j(\omega_0 + \phi_0)} + \frac{1}{2} A_0 e^{-j(\omega_0 + \phi_0)}$$

  - it can be written as the real part of an exponential signal:

$$s(t) = \Re\left\{A e^{j\omega_0 t}\right\}, \qquad A = A_0 e^{j\phi_0}$$

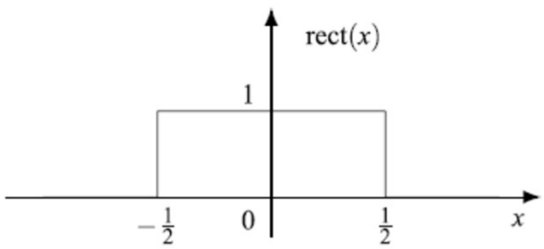
Some useful signals

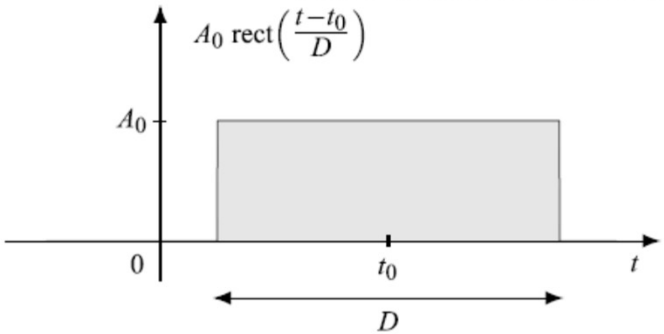
- The step signal: $s(t) = A_0 1(t - t_0),$
- Where the unit step function is:

$$1(x) = \begin{cases} 0, & \text{for } x < 0, \\ 1, & \text{for } x > 0. \end{cases} \qquad 1(0) = \frac{1}{2}$$

- The rectangular function:

$$\text{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2}, \\ 0, & \text{for } |x| > \frac{1}{2}, \end{cases}$$

- 



Some useful signals

- A triangular pulse: $$\mathrm{triang}(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1; \\ 0 & \text{for } |x| > 1. \end{cases}$$

- The impulse: $\delta(t)$ is assumed to vanish for $t \neq 0$

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1, \qquad \int_{-\infty}^{\infty} \delta(t) s(t)\,\mathrm{d}t = s(0).$$

  - Can be seen as a limit as D tends to zero.

$$r_D(t) = \frac{1}{D}\mathrm{rect}\left(\frac{t}{D}\right), \qquad \delta(t) = \lim_{D \to 0} r_D(t).$$

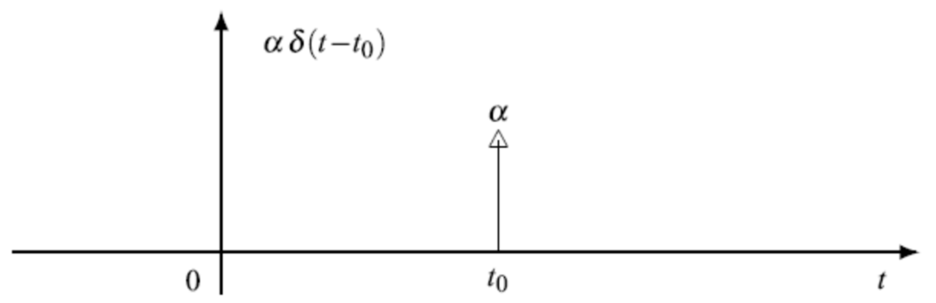
$$\lim_{D \to 0} \int_{-\infty}^{\infty} r_D(t) s(t)\,\mathrm{d}t = \lim_{D \to 0} \frac{1}{D} \int_{-D/2}^{D/2} s(t)\,\mathrm{d}t = s(0),$$

On the impulse

$$\int_{-\infty}^{\infty} s(t)\delta(t-t_0)\,\mathrm{d}t = \int_{-\infty}^{\infty} s(t+t_0)\delta(t)\,\mathrm{d}t = s(t_0).$$

$$\int_{-\infty}^{\infty} \delta(-t)s(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \delta(t)s(-t)\,\mathrm{d}t = s(0) = \int_{-\infty}^{\infty} \delta(t)s(t)\,\mathrm{d}t,$$

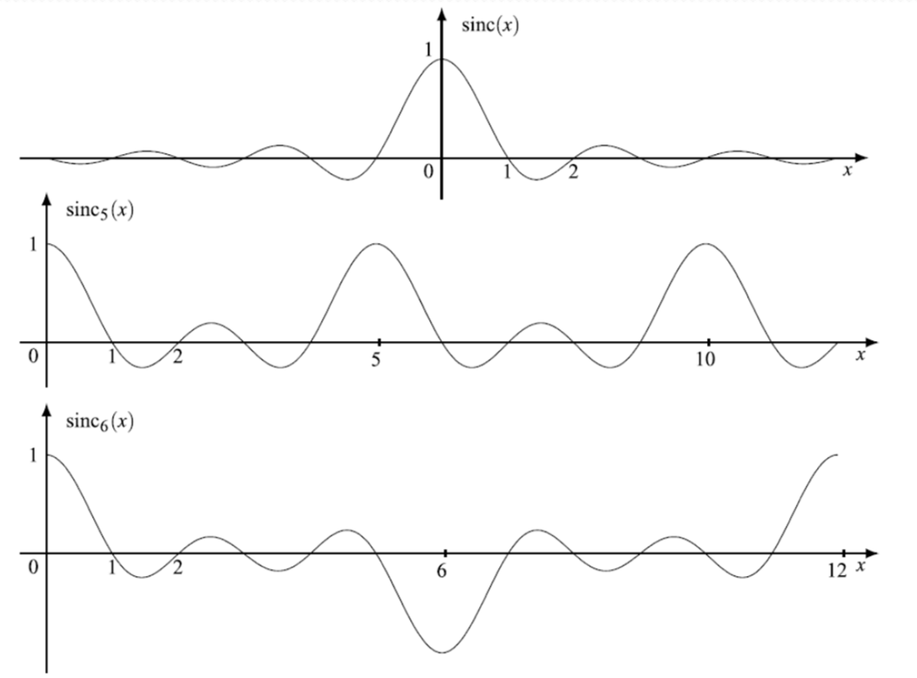
$$s(t) = \int_{-\infty}^{+\infty} s(u)\delta(t-u)\,\mathrm{d}u.$$



The sinc pulses

$$A_0 \operatorname{sinc}\left(\frac{t-t_0}{T}\right), \qquad \operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

- The periodic sinc $\operatorname{sinc}_N(x) = \dfrac{1}{N}\dfrac{\sin \pi x}{\sin \frac{\pi}{N} x},$



Convolution

- Given two continuous signals *x(t)* and *y(t)*, their convolution defines a new signal:

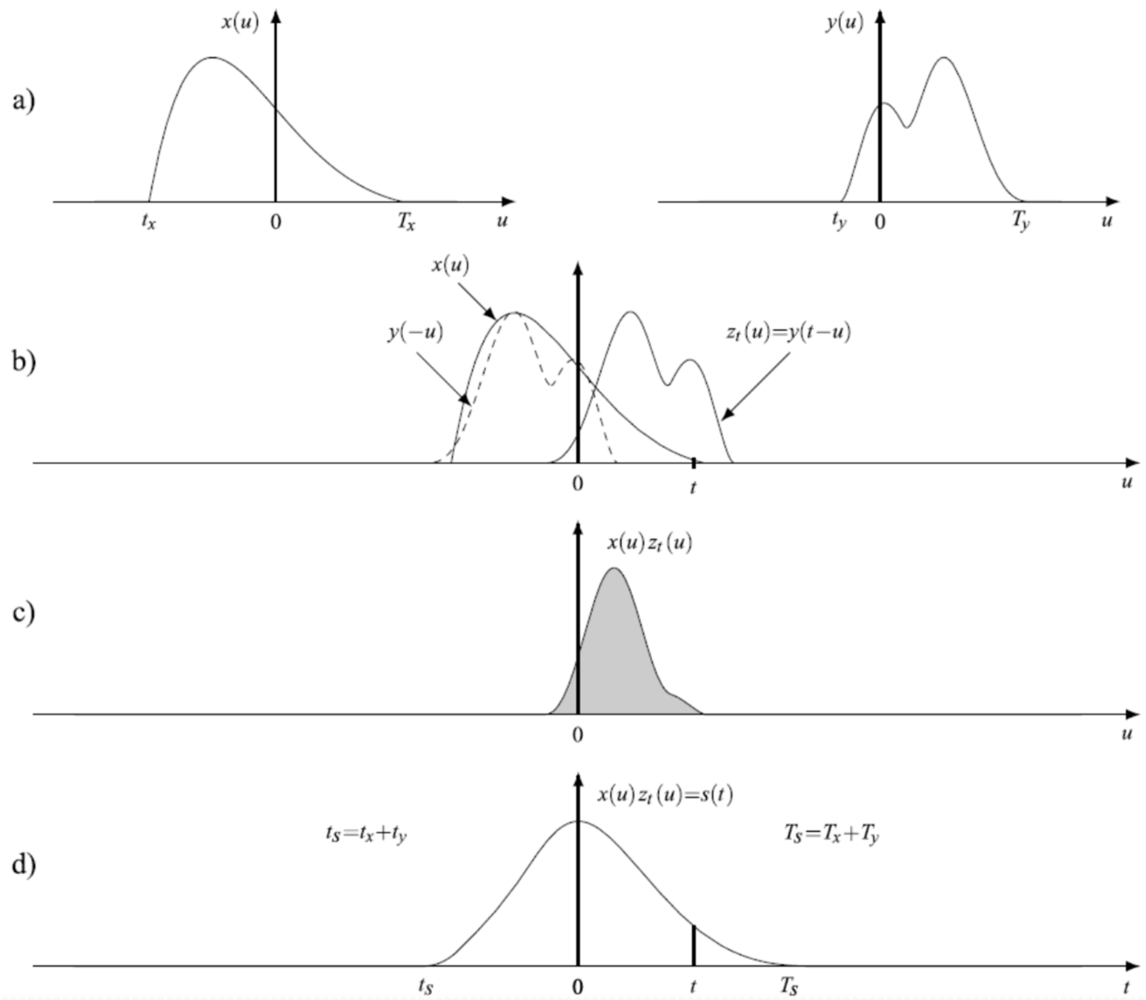
$$s(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)\,\mathrm{d}u.$$

- This is concisely denoted by: $s = x * y$
- If we define: $z_t(u) = z(u-t) = y\big(-(u-t)\big) = y(t-u),$

The convolution becomes: $s(t) = \int_{-\infty}^{+\infty} x(u)z_t(u)\,\mathrm{d}u.$

Convolution

In conclusion, to evaluate the convolution *at the chosen time t*, we multiply $x(u)$ by $z_t(u)$ and integrate the product.



Convolution

- In this interpretation, we hold the first signal while inverting and shifting the second.
- However, with a change of variable $v = t - u$, we obtain the alternative form

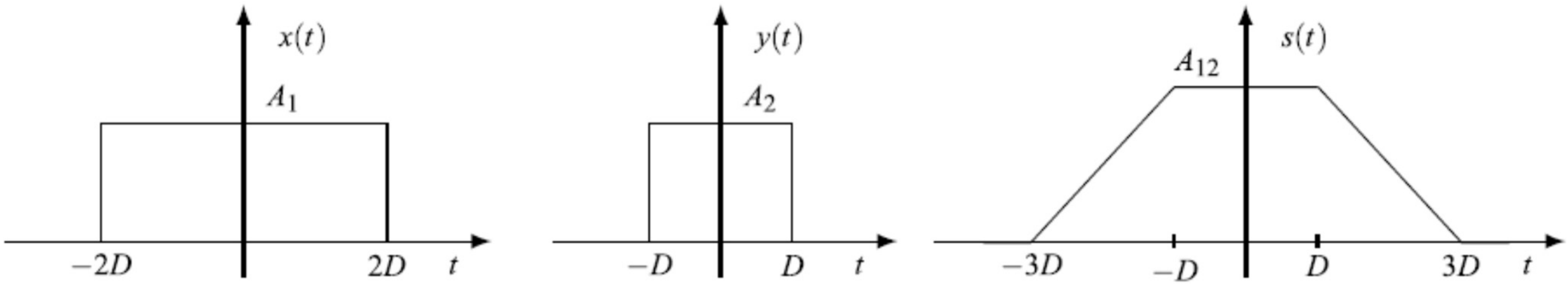
$$s(t) = \int_{-\infty}^{+\infty} x(t-u)\,y(u)\,\mathrm{d}u,$$

in which we hold the second signal and manipulate the first to reach the same result.

Convolution example

- We want to evaluate the convolution of the rectangular pulses

$$x(t) = A_1 \operatorname{rect}\left(\frac{t}{4D}\right), \qquad y(t) = A_2 \operatorname{rect}\left(\frac{t}{2D}\right).$$

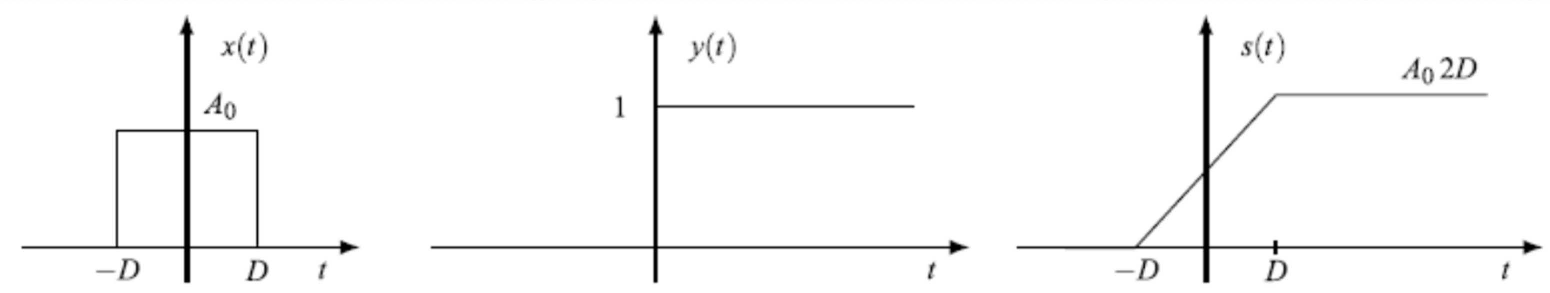


$$s(t) = \begin{cases} 0, & \text{if } t < -3D \text{ or } t > 3D; \\ A_1 A_2 (t + 3D), & \text{if } -3D < t < -D; \\ A_1 A_2 2D, & \text{if } -D < t < D; \\ A_1 A_2 (3D - t), & \text{if } D < t < 3D. \end{cases}$$

Convolution example

- We evaluate the convolution of the signals

$$x(t) = A_0 \operatorname{rect}\left(\frac{t}{2D}\right), \qquad y(t) = \mathrm{u}(t)..$$



$$s(t) = \begin{cases} 0, & \text{if } t < -D; \\ A_0(t+D), & \text{if } -D < t < D; \\ A_0 2D, & \text{if } t > D, \end{cases}$$

Convolution of a periodic signal

- The convolution of two periodic signals *x(t)* and *y(t)* with the **same period** T_p is then defined as:

$$x * y(t) \triangleq \int_{t_0}^{t_0+T_p} x(u)y(t-u)\,\mathrm{d}u.$$

- where the integral is over an arbitrary period (t_o, t_o+T_p). This form is sometimes called the *cyclic convolution* and then the previous form the *acyclic convolution*.

The Fourier Series

- We recall that in 1822 Joseph Fourier proved that an arbitrary (real) function of a real variable $s(t)$, $t \in \mathbb{R}$, having period T_p, can be expressed as the sum of a series of sine and cosine functions with frequencies multiple of the *fundamental frequency* $F = 1/T_p$, namely

$$s(t) = A_0 + \sum_{k=1}^{\infty} [A_k \cos 2\pi k F t + B_k \sin 2\pi k F t].$$

The exponential form

- A continuous signal $s(t)$, $t \in \mathbb{R}$, with period T_p, can be represented by the *Fourier series*

$$s(t) = \sum_{n=-\infty}^{\infty} S_n e^{i2\pi nFt}, \quad F = \frac{1}{T_p},$$

- Where:

$$S_n = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t) e^{-i2\pi nFt} \, dt, \quad n \in \mathbb{Z}.$$

Some properties of the Fourier Series

- Time shift:

$$x(t) = s(t - t_0) \longrightarrow X_n = S_n e^{-i2\pi n F t_0}.$$

- Mean Value:

$$m_s(T_p) = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t)\,\mathrm{d}t = S_0.$$

- Parseval's theorem:

$$P_s = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} \left|s(t)\right|^2 \mathrm{d}t = \sum_{n=-\infty}^{+\infty} |S_n|^2.$$

Examples

- A real sinusoid:

$$s(t) = A_0 \cos(2\pi f_0 t + \varphi_0) \longrightarrow s(t) = \frac{1}{2} A_0 e^{i\varphi_0} e^{i2\pi F t} + \frac{1}{2} A_0 e^{-i\varphi_0} e^{-i2\pi F t}.$$

$$\Downarrow$$

$$S_1 = \frac{1}{2} A_0 e^{i\varphi_0}, \qquad S_{-1} = \frac{1}{2} A_0 e^{-i\varphi_0}, \qquad S_n = 0 \quad \text{for } |n| \neq 1.$$

- A square wave:

$$s(t) = \sum_{n=-\infty}^{+\infty} A_0 \operatorname{rect}\left(\frac{t - nT_p}{dT_p}\right) = A_0 \operatorname{rep}_{T_p} \operatorname{rect}\left(\frac{t}{dT_p}\right), \quad 0 < d \leq 1$$

$$\Downarrow$$

$$S_n = \frac{1}{T_p} \int_{-\frac{1}{2} dT_p}^{\frac{1}{2} dT_p} A_0 e^{-i2\pi n F t} \, dt. \qquad S_n = S_0 \operatorname{sinc}(nd), \quad S_0 = A_0 d.$$

The Fourier Transform

- An aperiodic signal *s(t)*, $t \in \mathbb{R}$, can be represented by the *Fourier integral:*

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{i2\pi f t} \, df, \quad t \in \mathbb{R},$$

- *And*

$$S(f) = \int_{-\infty}^{+\infty} s(t) e^{-i2\pi f t} \, dt, \quad f \in \mathbb{R}.$$

$$s(t) \xrightarrow{\mathcal{F}} S(f), \quad S(f) \xrightarrow{\mathcal{F}^{-1}} s(t).$$

Interpretation

- In the Fourier series, a *continuous-time* periodic signal is represented by a *discrete frequency* function

$$S_n = S(nF).$$

- In the Fourier Transform, this is no more true and we find a symmetry between the time domain and the frequency domain, which are both continuous.
- In the Fourier Transform a signal is represented as the sum of infinitely many exponential functions of the form

$$[S(f)\,\mathrm{d}f]\mathrm{e}^{\mathrm{i}2\pi f t}, \quad f \in \mathbb{R}$$

Properties

- For real signals the Fourier Transform has the Hermitian Symmetry:

$$S(-f) = S^*(f),$$

- Time shift:

$$s(t - t_0) \xrightarrow{\mathcal{F}} S(f)e^{-i2\pi f t_0}$$

- Frequency shift:

$$S(f - f_0) \xrightarrow{\mathcal{F}^{-1}} s(t)e^{i2\pi f_0 t}$$

- Convolution:

$$x(t) * y(t) \xrightarrow{\mathcal{F}} X(f)Y(f)$$

- Product:

$$x(t)y(t) \xrightarrow{\mathcal{F}} X(f) * Y(f)$$

Examples

- Rectangular pulse and sinc function

$$S(f) = A_0 \int_{-\frac{1}{2}D}^{\frac{1}{2}D} \mathrm{e}^{-\mathrm{i}2\pi f t}\,\mathrm{d}t = \frac{A_0}{-\mathrm{i}2\pi f}\left(\mathrm{e}^{-\mathrm{i}\pi f D} - \mathrm{e}^{\mathrm{i}\pi f D}\right) = A_0 \frac{\sin \pi f D}{\pi f}.$$

$$A_0\,\mathrm{rect}(t/D) \xrightarrow{\mathcal{F}} A_0 D\,\mathrm{sinc}(f D).$$

$$S(t) = A_0 D\,\mathrm{sinc}(t D) \xrightarrow{\mathcal{F}} s(-f) = A_0\,\mathrm{rect}(-f/D),$$

- Impulses

$$S(f) = \int_{-\infty}^{+\infty} \delta(t - t_0)\mathrm{e}^{-\mathrm{i}2\pi f t}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{i}2\pi f t_0}.$$

$$\delta(t - t_0) \xrightarrow{\mathcal{F}} \mathrm{e}^{-\mathrm{i}2\pi f t_0}$$

Examples

- Periodic signals

$$\cos 2\pi Ft = \frac{1}{2}\left(e^{i2\pi Ft} + e^{-i2\pi Ft}\right) \xrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f-F) + \delta(f+F)\right],$$

$$\sin 2\pi Ft = \frac{1}{2i}\left(e^{i2\pi Ft} - e^{-i2\pi Ft}\right) \xrightarrow{\mathcal{F}} \frac{1}{2i}\left[\delta(f-F) - \delta(f+F)\right].$$

$$s(t) = \sum_{n=-\infty}^{+\infty} S_n e^{i2\pi nFt} \xrightarrow{\mathcal{F}} \sum_{n=-\infty}^{+\infty} S_n \delta(f-nF).$$

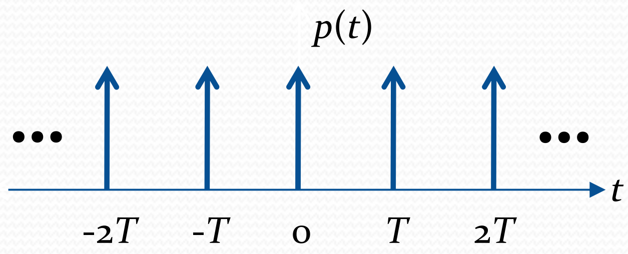
- Signum signal

$$\operatorname{sgn}(t) \xrightarrow{\mathcal{F}} \frac{1}{i\pi f}.$$

- Step signal

$$1(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) \xrightarrow{\mathcal{F}} \frac{1}{2}\delta(f) + \frac{1}{i2\pi f}.$$

Representation of a CT Signal Using Impulse Functions



Recall our expression for a pulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

A sampled version of a CT signal, x(t), is:

$$x_s(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta\big(t - nT\big) = \sum_{n=-\infty}^{\infty} x(nT)\delta\big(t - nT\big)$$

This is known as idealized sampling.

We can derive the complex Fourier series of a pulse train:

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t} \quad \text{where} \quad \omega_0 = 2\pi / T$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-ik\omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-ik\omega_0 t}\,dt = \frac{1}{T}\Big[e^{-ik\omega_0 t}\Big]_{t=0} = \frac{1}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{ik\omega_0 t}$$

Fourier Transform of a Sampled Signal

The Fourier series of our sampled signal, $x_s(t)$ is:

$$x_s(t) = p(t)x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_0 t}$$

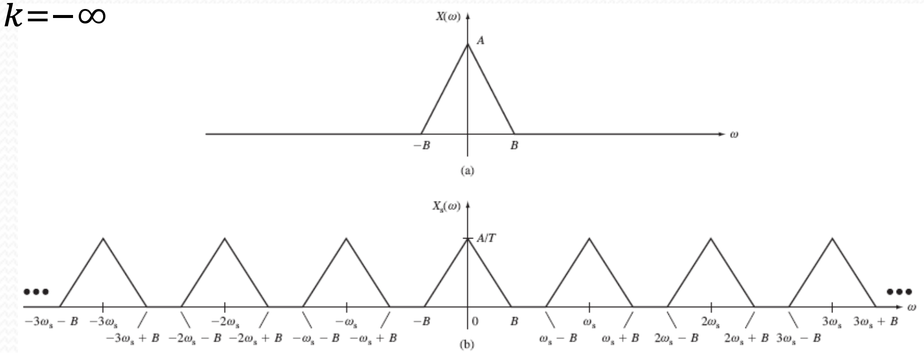
Recalling the Fourier transform properties of linearity (the transform of a sum is the sum of the transforms) and modulation (multiplication by a complex exponential produces a shift in the frequency domain), we can write an expression for the Fourier transform of our sampled signal:

$$X_s\left(e^{j\omega}\right) = \mathcal{F}\{p(t)x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_0 t}\right\} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathcal{F}\left\{x(t) e^{jk\omega_0 t}\right\} =$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(e^{j(\omega - k\omega_0)}\right)$$

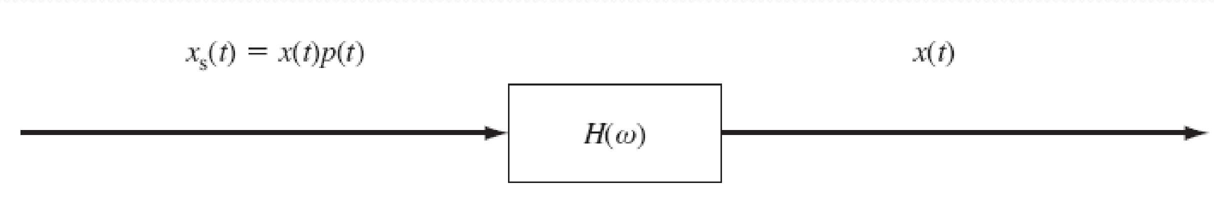
If our original signal, *x(t)*, is bandlimited:

$$\left|X(e^{j\omega})\right| = 0 \quad \text{for} \quad \omega > B$$

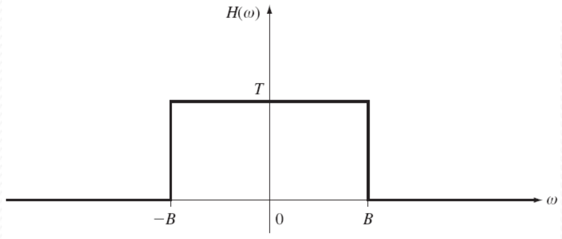


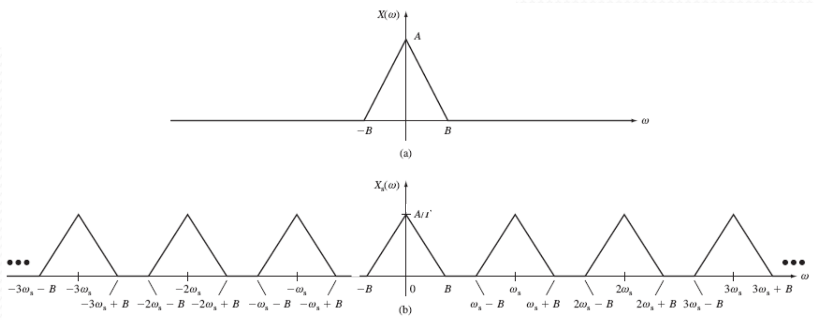
Signal Reconstruction

Note that if $\omega_s \geq 2B$, the replicas of $X\left(e^{j\omega}\right)$ do not overlap in the frequency domain. We can recover the original signal exactly.



Lowpass filter





The sampling frequency, $\omega_s = 2B$, is referred to as the Nyquist sampling frequency.

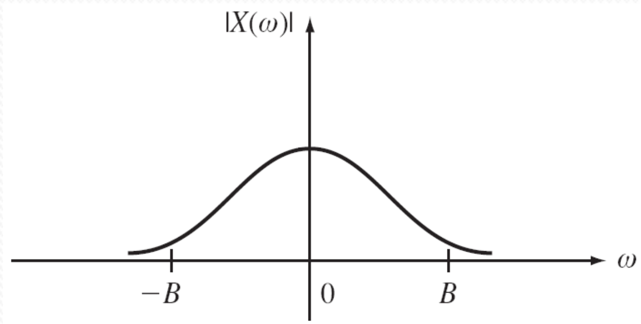
There are two practical problems associated with this approach:

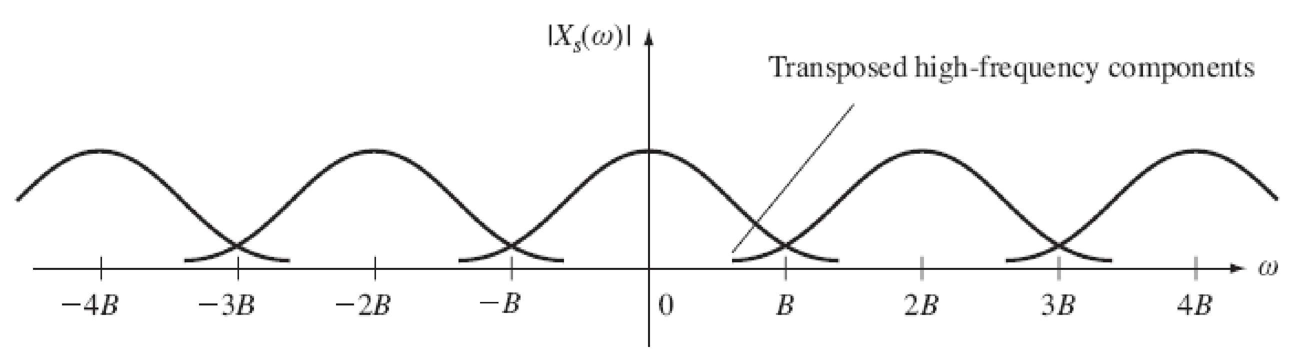
- The lowpass filter is not physically realizable.
- The input signal is typically not bandlimited.

Aliasing

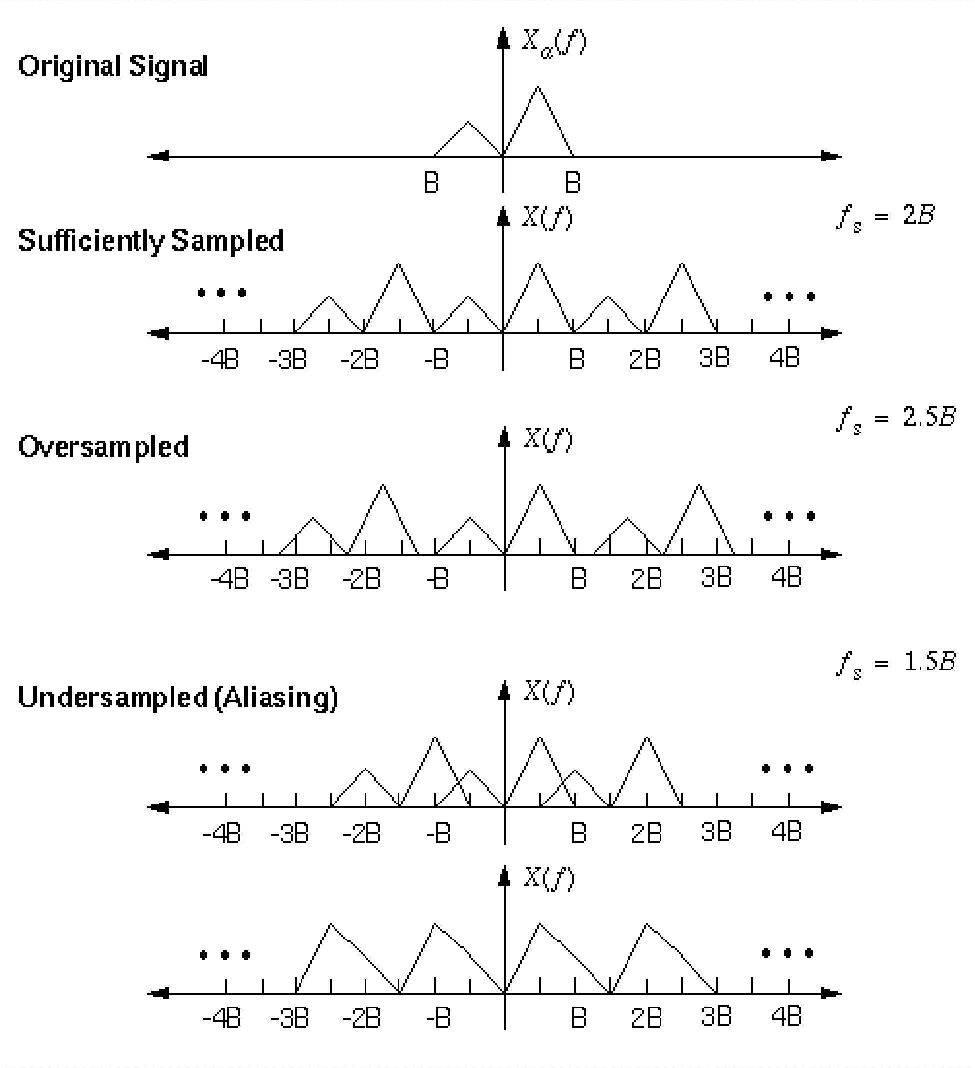
Recall that a time-limited signal cannot be bandlimited. Since all signals are more or less time-limited, they cannot be bandlimited. Therefore, we must lowpass filter most signals before sampling. This is called an anti-aliasing filter and are typically built into an analog to digital (A/D) converter.

If the signal is not bandlimited distortion will occur when the signal is sampled. We refer to this distortion as aliasing:





Undersampling and Oversampling of a Signal

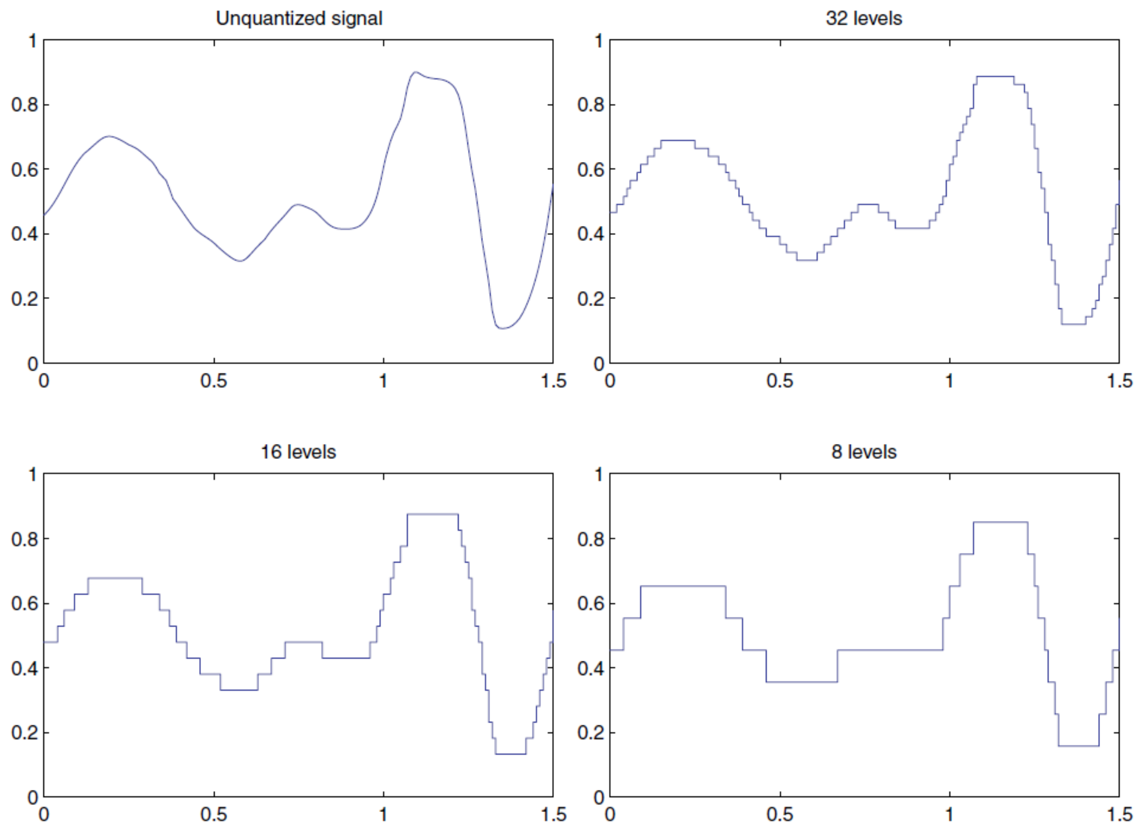


Quantization

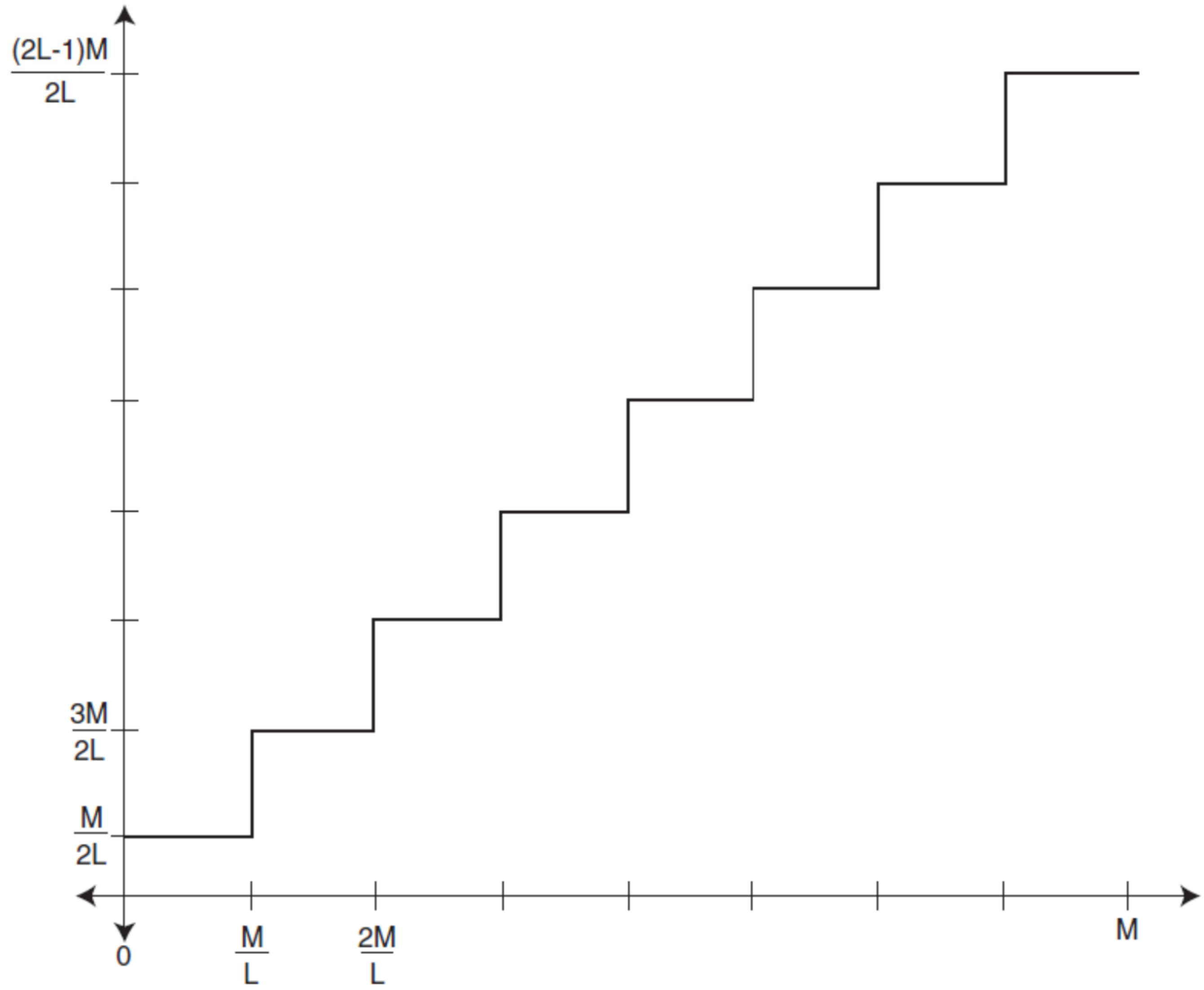
- Quantization makes the range of a signal discrete, so that the quantized signal takes on only a discrete, usually finite, set of values.
- Unlike sampling (where we saw that under suitable conditions exact reconstruction is possible), quantization is generally irreversible and results in loss of information.
- It therefore introduces distortion into the quantized signal that cannot be eliminated.

Quantization

- With L levels, we need $N = \log_2 L$ bits to represent the different levels,
- conversely, with N bits we can represent $L = 2^N$ levels.



Uniform quantization



Uniform quantization

256 levels

32 levels

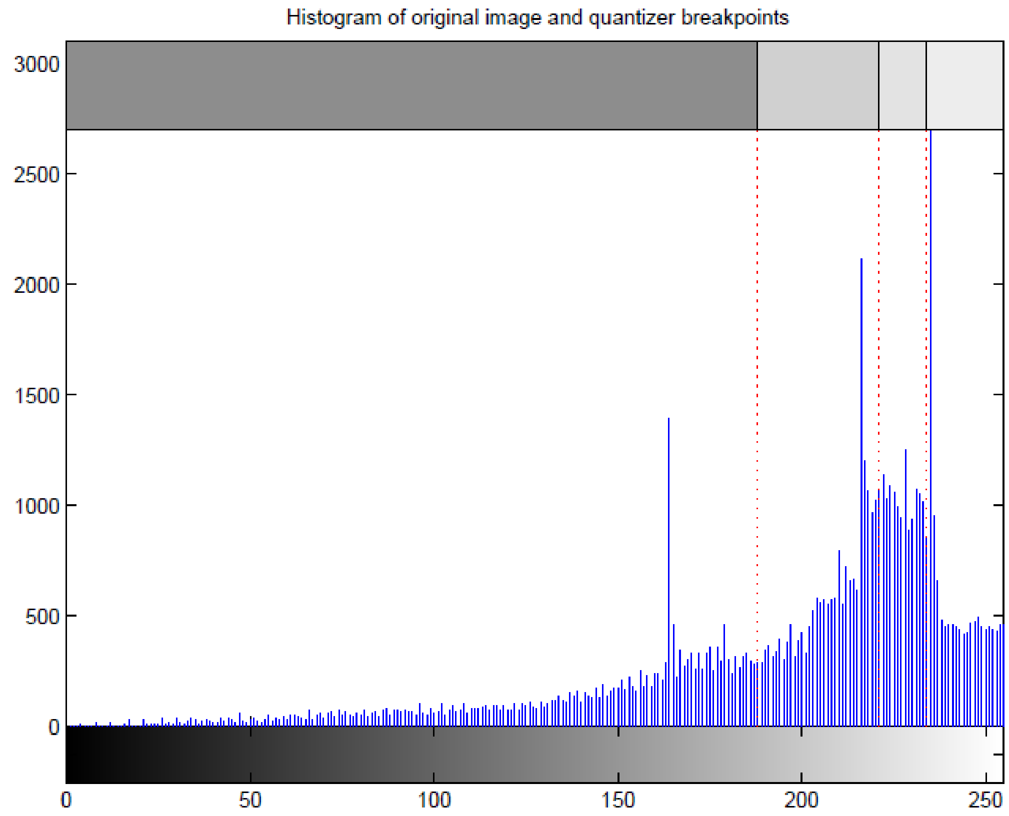
16 levels

8 levels

4 levels

2 levels

Non uniform quantization



Uniform vs. non-uniform quantization

Uniform quantization, 4 levels

Non−uniform quantization, 4 levels